THE ART AND SCIENCE

OF

DUAL ARITHMETIC.

DUAL ARITHMETIC;

A NEW ART.

PART THE SECOND.

THE DESCENDING BRANCH OF THE "*ART*" AND THE "*SCIENCE*" OF DUAL ARITHMETIC.

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THE ART AND SCIENCE

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DUAL ARITHMETIC.

to clumsy and restricted treatment, and to show that Dual Arithmetic harmonized not only with Common Arithmetic, but also with Algebra, and the higher branches of Mathematics.

II.

In the present work I have developed the descending branch of the system, connected it with the ascending, and treated Dual Arithmetic as a science. Thus, I have completed the art and science of a calculus of the concrete values of quantities known or unknown, and shown for the first time how all mathematical functions, direct and inverse, can be submitted to the operations of Dual Arithmetic without the aid of Tables. Among the several operations and their converse that can be done with the greatest ease by direct processes of Dual Arithmetic, without the aid of Tables, in an endless variety of ways, I shall only mention here, the Involution, and the Evolution of numbers for any root or power; the direct calculation of the logarithm of any number whatever to any base; and the general methods of determining numerical roots of all orders of equations and also of exponential and transcendental equations, whether the bases be known or unknown.

III.

"The Young Dual Arithmetician" is a work designed to qualify young Students to read and understand the larger works, to render the practical calculator independent of tables

of common logarithms, and to demonstrate that, if tables be preferred, those of dual logarithms are incomparably the best. Any schoolboy may construct an extensive table of dual logarithms in an incredibly short space of time, and afterwards test its accuracy at any point, if he only understands Common Addition and Subtraction.

IV.

"The Dual Doctrine of Angular Magnitudes and Functions, and its Application to Plane and Spherical Trigonometry."

In this work, trigonometry is treated in an original and philosophical manner by demonstrating the transcendental formulæ of trigonometry without the aid of impossible quantities.

V.

"Tables of Ascending and Descending Dual Numbers, Dual Logarithms, and their corresponding Natural Numbers; and of Angular Magnitudes."

When operations are performed with dual numbers in their lowest terms, and tables are used, it is not necessary that such tables should range beyond the natural numbers from ·41421356I to I·, and from I· to ·70710678 (*see* the present work, p. 12). But this volume of Tables exceeds these limits, and is more comprehensive, and more easily used than any hitherto calculated. These tables are equal in power to Babbage's and Callet's combined, and take up less than one-eighth part of their space. Dual Tables ranging

from 1· to 2·99161136, ascending, and from 1· to ·299161136, descending, have the greatest power in economizing the time and labour of the calculator.

I intend now to turn my attention to developing another new mathematical science which I have discovered, and styled the "Calculus of Form." It establishes modern analysis on a purely mathematical basis, and rejects the reasoning of the Differential and other methods now current.

OLIVER BYRNE.

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THE

ART AND SCIENCE

OF

DUAL ARITHMETIC.

INTRODUCTION.

ON THE ASCENDING AND DESCENDING BRANCHES OF DUAL ARITHMETIC WITH EXTENSION OF THE NOTATION.

In the work entitled "Dual Arithmetic, a New Art," and in its reissue with an analysis, we showed that any number whatever, whether great or small, might be reduced to the form

$$2^n\, 10^m \downarrow u_1,\ u_2,\ u_3,\ \&c.$$

where this notation is used for the continued product

$$2^n \times 10^m \times (1{\cdot}1)^{u_1} \times (1{\cdot}01)^{u_2} \times (1{\cdot}001)^{u_3} \times \&c.$$

Thus

$$2^2\, 10^3 \downarrow 3,1,4,1,2,1,1,3, = 2^2 \times 10^3 \times (1{\cdot}1)^3\, (1{\cdot}01)^1\, (1{\cdot}001)^4\, (1{\cdot}0001)^1$$
$$(1{\cdot}00001)^2\, (1{\cdot}000001)^1\, (1{\cdot}0000001)^1\, (1{\cdot}00000001)^3$$
$$= 2^2 \times 10^3 \times 1{\cdot}34985881 = 5399{\cdot}43524.$$

The transformation of any common number into a dual number, and the converse operation were fully shown in that work.

It was also shown how any of the digits of any dual number might be transformed into zero, the remaining digits being altered in value.

When the first seven digits were so transformed, the eighth remaining digit was called the ultimate value of the dual number in the eighth position, and was shown to possess all the properties of a common logarithm of eight places of decimals.

And then

$$2^n\ 10^m \downarrow u_1,\ u_2,\ u_3,\ u_4,\ u_5,\ u_6,\ u_7,\ u_8, = 2^n\ 10^m \downarrow 0,0,0,0,0,0,0,U,$$

was written

$$2^n\ 10^m \overset{8}{\downarrow} U,.$$

The method of calculating these ultimate values for every dual digit, as well as for the common numbers 2 and 10, was shown.

By this means, every arithmetical operation requiring the use of logarithms was performed without the use of tables, and by methods involving only the simplest processes of arithmetic.

Arithmetical solutions of many problems and their converse were obtained by using the dual arithmetic in its simplest form of development, which have defied the skill of previous investigators, with all the aids of the highest forms of calculus.

It was also shown how the dual system of calculation blended with the operations of common arithmetic without interfering with the generality of either.

This was done under great disadvantages, as it was inexpedient at first to introduce into the art more than one of its branches.

The student who understands what has been written and published on this subject is now in a position to enter into the more extended development of the subject.

Since any number may be represented in the form

$$N = 2^n\, 10^m \downarrow u_1,\, u_2,\, u_3,\, u_4,\, u_5,\, \&c.$$

we may omit the bases 2 and 10, with as much advantage in perspicuity as we omitted the bases 1·1, 1·01, 1·001, &c. and write the above expression in the form

$$N = {}^m\!\!\downarrow^n\ u_1,\, u_2,\, u_3,\, u_4,\, \&c.$$

By using digits to the left of the arrow, the powers of 10 may be dispensed with altogether.

Where

$$N = w_3,\, w_2,\, w_1,\ \downarrow^n\ u_1,\, u_2,\, u_3,\, u_4,\, \&c.$$

is a notation for a continued product of the form

$$N = (1 + 1000)^{w_3}\, (1 + 100)^{w_2}\, (1 + 10)^{w_1}\, (1 + 1)^n\, (1 + \cdot 1)^{u_1}$$
$$(1 + \cdot 01)^{u_2}\, (1 + \cdot 001)^{u_3}\, (1 + \cdot 0001)^{u_4}\ \&c.$$

Now

$$(1 + 1000)^{w_3} = 10^{3w_3}\left\{1 + \frac{1}{10^3}\right\}^{w_3} = (10^3)^{w_3} \overset{3}{\downarrow}_{w_3},$$

$$(1 + 100)^{w_2} = 10^{2w_2}\left\{1 + \frac{1}{10^2}\right\}^{w_2} = (10^2)^{w_2} \overset{2}{\downarrow}_{w_2},$$

$$(1 + 10)^{w_1} = 10^{w_1}\left\{1 + \frac{1}{10}\right\}^{w_1} = (10)^{w_1} \overset{1}{\downarrow}_{w_1},$$

Hence,

$$N = \quad w_3,\, w_2,\, w_1,\ \downarrow^n\ u_1,\, u_2,\, u_3,\, u_4,\ \&c.$$
$$= 10^{w_1 + 2w_2 + 3w_3}\ \downarrow^n\ (u_1 + w_1),\, (u_2 + w_2),\, (u_3 + w_3),\, u_4\ \&c.$$
$$= \quad {}^{(w_1 + 2w_2 + 3w_3)}\!\downarrow^n\ (u_1 + w_1),\, (u_2 + w_2),\, (w_3 + w_3),\, u_4\ \&c.$$

This last expression shows how powers of 10 in a dual number may be replaced by digits on the left of the arrow, such digits representing powers of the bases 11, 101, 1001, &c. increasing from left to right.

And conversely how digits on the left of the arrow may be transferred to the right.

Thus,

$$,3,4,2 \downarrow^{3} 5,6,7,8,9, = {}^{10}\downarrow^{3} (5+2),(4+6),(7+3),8,9, = {}^{10}\downarrow^{3} 7,10,10,8,9,$$

Again,

$${}^{11}\downarrow^{5} 3,5,6,8, = 4,3, \downarrow^{5} (3-3),(5-4),6,8, = 4,3, \downarrow^{5} 0,1,6,8,$$

or,

$${}^{11}\downarrow^{5} 3,5,6,8, = 3,0,2, \downarrow^{5} (3-2),(5-0),(6-3),8 = 3,0,2, \downarrow^{5} 2,5,3,8,$$

In writing the power of **2**, which is always at the middle of the arrow, care must be taken not to confound it with the figure at the top of the arrow, to the right, designating the position of the first dual digit following it.

The comma (,) is employed in the operations of dual arithmetic, while the period (·) is retained to separate whole numbers from decimal fractions; this part of the general notation should be remembered.

It will be found that the comma accompanying a dual digit, or a dual logarithm, will be a sufficient distinguishing characteristic without employing the strong black figure, as above, and in the Work previously published.

Thus,

${}^{7}\overset{4}{\downarrow}{}^{2}$ 5,6,7,8 is a short expression for $(10)^{7}\,(2)^{4}_{?}$ $\downarrow$ 0,0,0,5,6,7,8,

And

$${}^{8}\overset{3}{\downarrow}{}^{6} \text{ 1,2,3,4 for } (10)^{8}\,(2)^{6} \downarrow 0,0,1,2,3,4,$$

In the Work just referred to, we showed that any dual number might be transformed into another, any number of

whose digits, counting from the first, might be zero, the next remaining digit being increased in value.

The extent to which this reduction was necessary to be carried in practice was shown to depend upon the accuracy of the arithmetical result required to be obtained.

Thus for results true to four places of figures, it was shown that dual numbers of only five digits were required, and that when four of these were reduced to zero, the fifth gave all the properties of a common logarithm of five places of decimals, &c.

For results true to seven places of figures, eight dual digits are required, and when the first seven of these are reduced to zero, the eighth is called a dual logarithm. See "Dual Arithmetic, a New Art," pp. 27, 28.

It was shown also that there were several values which give correct results in these fifth, sixth, seventh and eighth positions, but a particular set of values were shown which were termed ultimate values.

For calculating these ultimate values, as well as for an account of their properties, we must refer to "Dual Arithmetic, a New Art," pp. 212—214.

In the generalization of the form of the dual number ascending branch

Where

$$N = {}^{m}\!\downarrow^{n}\ u_1,\ u_2,\ u_3,\ u_4,\ \&c.$$

is written under the form

$$N = w_3,\ w_2,\ w_1,\ \downarrow^{n}\ u_1,\ u_2,\ u_3,\ u_4,\ \&c.$$

We must remember that m represents a positive whole number.

Or

$$N = (1 + 1000)^{w_3}\ (1 + 100)^{w_2}\ (1 + 10)^{w_1}\ (1 + 1)^{n}\ (1 + \cdot 1)^{u_1}$$
$$(1 + \cdot 01)^{u_2}\ (1 + \cdot 001)^{u_3}\ (1 + \cdot 0001)^{u_4}\ \&c.$$

It is evident by inspecting the form of this continued product, that by means of the powers of these bases written to the right and left of the arrow, including that of the base (1 + 1) or 2 on the arrow, that any number from + infinity to 1, can be expressed to any degree of accuracy. The bases being neglected in the representation of the quantities just as powers of 10 are neglected in ordinary arithmetic, and the bases in logarithmic arithmetic.

Quantities less than 1 are represented under the form $\mathrm{N} = {}^{m}\!\downarrow^{n} u_1, u_2, u_3, u_4,$ &c. by making m negative, but in that case we do not transfer m to dual digits on the left of the arrow.

A more complete method of representing numbers less than 1 will be shown when we discuss the notation of the descending branch of dual arithmetic.

DUAL LOGARITHMS, ASCENDING BRANCH.

$$\mathrm{U} = \overset{8}{\downarrow} u,$$

Here this notation signifies that u is the ultimate value of the common number U in the eighth position, or we may say that u is the dual logarithm of the common number U.

As results true to seven places of figures are those most commonly used in arithmetical operations, when we speak of a dual logarithm, without specifying position, we regard it as of the eighth position.

As the use of the logarithm of a number is most frequent in symbolical operations, instead of writing

$$\text{Dual log. U} = u, \text{ when } \mathrm{U} = \overset{8}{\downarrow} u \text{ we write } \downarrow_{8}(\mathrm{U}) = u,$$

Thus since

$$2 = \overset{8}{\downarrow} 69314718,$$

$\therefore$ $\downarrow,(2) = 69314718,$ Or dual log. of $2 = 69314718$, a whole number.

Hence it will be seen that by attaching a comma to the sign $\downarrow,$ we indicate an operation the exact converse of that represented by the sign $\overset{8}{\downarrow}$.

It is often necessary, when using a dual number not reduced to its ultimate position (but which can always be so reduced) to indicate that its logarithm is to be taken.

Thus supposing we have to indicate the logarithm of the number represented by the dual number

$$\downarrow 7,2,6,0,7,8,2,6,$$

Using the analogous notation to that above, we should write it

$$\downarrow, 7,2,6,0,7,8,2,6,$$

But

$$\downarrow 7,2,6,0,7,8,2,6, = \overset{8}{\downarrow} 69314718,$$

$$\therefore \quad \downarrow, 7,2,6,0,7,8,2,6, = 69314718$$

Hence on the whole, if

$$2 = \downarrow 7,2,6,0,7,8,2,6, = \overset{8}{\downarrow} 69314718,$$

Then

$$\downarrow,(2) = 69314718 = \downarrow, 7,2,6,0,7,8,2,6,$$

And taking away the commas attached to the arrows which indicate logarithms, we have

$$2 = \downarrow 7,2,6,0,7,8,2,6$$

This gives all the notation we require at present for logarithmic operations by the ascending branch.

NOTATION AND EXPLANATION OF THE DESCENDING BRANCH OF DUAL ARITHMETIC.

Any number N may be written as a continued product of the form

$$10^m \times (1 - \cdot 1)^{v_1} (1 - \cdot 01)^{v_2} (1 - \cdot 001)^{v_3} (1 - \cdot 0001)^{v_4} \text{ \&c.}$$

or $$10^m (\cdot 9)^{v_1} (\cdot 99)^{v_2} (\cdot 999)^{v_3} (\cdot 9999)^{v_4} \text{ \&c.}$$

In analogy with the notation used in the ascending branch of dual arithmetic, this continued product may be written thus

$$'v_1'v_2'v_3'v_4'v_5'v_6 \downarrow^m$$

where any of the digits v_1, v_2, v_3, &c. as well as m may be positive or negative.

Negative digits are only used when the descending branch is not combined with the ascending.

As in the ascending branch the power of 10, m, may be taken off the arrow and digits placed to the right when m is a + whole number.

Thus $$'v_1'v_2'v_3'v_4' \uparrow t_1't_2't_3' \text{ \&c.}$$

represents the continued product

$$(\cdot 9)^{v_1} (\cdot 99)^{v_2} (\cdot 999)^{v_3} (\cdot 9999)^{v_4} (9)^{t_1} (99)^{t_2} (999)^{t_3}$$

but the bases 9, 99, 999, or the digits t_1, t_2, t_3, &c. are seldom employed except in analytical inquiries.

For the most part this descending branch is only used in combination with the ascending one.

When so used, positive digits are only employed, and then the descending branch gives a method of converting all numbers

between 1 and minus infinity into dual numbers. When this combination is used, only + digits of both branches are required.

Thus any positive whole number between 0 and + infinity, may be represented under the form

$$'v_1\ 'v_2\ 'v_3 \uparrow w_3, w_2, w_1, \downarrow^n u_1, u_2, u_3, u_4, \text{ \&c.}$$

which represents the continued product

$$(\cdot 9)^{v_1} (\cdot 99)^{v_2} (\cdot 999)^{v_3} (1 + 1000)^{w_3} (1 + 100)^{w_2} (1 + 10)^{w_1} (1 + 1)$$
$$(1 + \cdot 1)^{u_1} (1 + \cdot 01)^{u_2} (1 + \cdot 001)^{u_3} \text{ \&c.}$$

This gives the power of representing any number, however small or however great, by the combination of the two branches, using only + digits.

For analytical purposes, it is necessary to extend the bases of the descending branch, so as to express zero and quantities which are negative of any magnitude whatever in that base.

The scheme of the bases of the descending branch may be written under this form.

$$-\infty, \ldots. (1 - 1000)^{x_4} (1 - 100)^{x_3} (1 - 10)^{x_2} (1 - 1)^{x_1} (1 - \cdot 1)^{v_1}$$
$$(1 - \cdot 01)^{v_2} (1 - \cdot 001)^{v_3} \text{ \&c. to } 1.$$

The base continually approaching to + 1 but never exceeding it.

Also, the descending bases may be employed under the form

$$+\infty \ldots. (1000 - 1)^{s_3} (100 - 1)^{s_2} (100 - 1)^{s_1} (1 - 1)^{x_1}$$
$$(\cdot 1 - 1)^{z_1} (\cdot 01 + 1)^{z_2} (\cdot 001 - 1)^{z_3} \text{ \&c. } \ldots\ldots. - 1.$$

The negative extensions of the bases, however, being solely used for analytical investigations, the base (1 – 1) as well as (1 – 10) (1 – 100), &c. (·1 – 1) (·01 – 1) (·001 – 1) &c. are not used in the present work.

Using the processes of operating on a number by ascending dual numbers,

$$
\begin{array}{l}
\cdot 9 \,|\, 9 \\
\hline
\cdot 99 \,|\, 99 \quad = \cdot 9 \downarrow 1, \\
\hline
\cdot 9999 \,|\, 9999 \quad = \cdot 9 \downarrow 1,1, \\
\hline
\cdot 99999999 \,|\, 99999999 \quad = \cdot 9 \downarrow 1,1,0,1, \\
\hline
\cdot 9999999999999999 \quad = \cdot 9 \downarrow 1,1,0,1,0,0,0,1,
\end{array}
$$

Therefore we may say that $\cdot 9 \downarrow 1,1,0,1,0,0,0,1, = 1$, very nearly.

Similarly, it may be shown that

$$\cdot 99 \downarrow 0,1,0,1,0,0,0,1, = 1$$

$$\cdot 999 \downarrow 0,0,1,0,0,1,0,0, = 1$$

$$\cdot 9999 \downarrow 0,0,0,1,0,0,0,1, = 1$$

Or

$$'1 \updownarrow 1,1,0,1,0,0,0,1, = 1$$

$$'1 \updownarrow_{2} 0,1,0,1,0,0,0,1, = 1$$

$$'1 \updownarrow_{3} 0,0,1,0,0,1,0,0, = 1$$

$$'1 \updownarrow_{4} 0,0,0,1,0,0,0,1, = 1$$

THE

SCIENCE OF DUAL ARITHMETIC,

AND THE

APPLICATION OF THE ART,

INVOLVING BOTH BRANCHES.

CHAPTER I.

THE GENERAL NOTATION APPLIED TO PARTICULAR NUMERICAL EXAMPLES, WITH SHORT METHODS FOR CONVERTING A NATURAL NUMBER TO A DUAL NUMBER; AND A DUAL NUMBER TO A DUAL LOGARITHM; AND *vice versâ.*

1. DUAL Arithmetic is *a new art of manœuvring numbers*, and also *a new science by which the relations of quantities are investigated* with ease and accuracy, with or without the use of tables.

2. The term *Dual* is employed because the art has *two branches*, the basis of each branch being composed of *two parts*, and because the digits of a dual number may be subjected to a variety of changes in magnitude and position, while at the same time the dual number remains equal in value to *two unchangeable extremes*, namely, a natural number, and a logarithm to a known base.

BASES OF THE ASCENDING BRANCH.

$$\text{(A)}\ \overset{\text{Limit}}{+\infty} \ldots\ldots (10001);\ (1001);\ (101);\ (11);\ (2);\ (1\cdot1);\ (1\cdot01);$$
$$(1\cdot001);\ (1\cdot0001);\ \ldots\ldots\ \overset{\text{Limit}}{1}$$

BASES OF THE DESCENDING BRANCH.

$$\text{(B)}\ \overset{\text{Limit}}{-\infty} \ldots (-999\dot{9});\ (-99\dot{9});\ (-9\dot{9});\ (-\dot{9});\ (0);\ (\cdot9);\ (\cdot99);$$
$$(\cdot999);\ \ldots\ldots\ \overset{\text{Limit}}{1}$$

Or

$$\text{(C)}\ \overset{\text{Limit}}{+\infty} \ldots (99\dot{9});\ (9\dot{9});\ (\dot{9});\ (0);\ (-\cdot9);\ (-\cdot99);\ (-\cdot999);\ \ldots\ \overset{\text{Limit}}{-1}$$

3. The sum of the bases (A) and (B) similarly circumstanced assume the values

$$0 \ldots\ldots (\dot{2});\ (\dot{2});\ (\dot{2});\ (\dot{2});\ \ldots\ldots\ 2.$$

Of (A) and (C)

$$+\infty \ldots (200\dot{0});\ (20\dot{0});\ (2\dot{0});\ (\dot{2});\ (\cdot2);\ (\cdot02);\ (\cdot002);\ \ldots\ 0$$

Of (B) and (C)

$$0 \ldots\ldots (0);\ (0);\ (0);\ (0);\ (0);\ \ldots\ldots\ 0$$

The descending branch, and the ascending, and both combined may be represented respectively by the three following general symbols.

$$'v\ \ {}_{n}\!\underset{p}{\uparrow}\!{}_{m} \qquad\qquad {}_{m}\!\overset{p}{\downarrow}\!{}_{n}\ \ u,$$

$$'v\ \ {}_{m}\!\overset{p}{\underset{p}{\updownarrow}}\!{}_{n}\ \ u,$$

4. A small figure placed at p designates the position occupied by a dual digit, and sometimes points out the leading position occupied by the first of more dual digits than one.

$$m \text{ expresses } 10^m$$

$$\overline{m} \quad ,, \quad \frac{1}{10^m}$$

$$n \quad ,, \quad 2^n$$

$$\overline{n} \quad ,, \quad \frac{1}{2^n}$$

A dual number of positive dual digits has always an exact value in common numbers when no contractions are employed in the reduction. When eight positions to the right and eight to the left of the signs ↑ ↓, counting from left to right in both cases, are occupied by ciphers or other digits, the sign ↓ being placed before the eight ascending digits, *on the left*, and ↑ after the eight descending, *on the right;* yet with respect to range the dual number is said to be one of eight digits although sixteen positions, and other position between the signs ↑ and ↓ may be occupied. If one of the signs ↑ or ↓ is omitted, the positions attached to it are supposed to be occupied by ciphers.

5. When the last dual digit, and all that follow, are rejected, and when the last is 5, 6, 7, 8, or 9, the digit preceding may be counted one more, as in decimal arithmetic.

$$2^{\cdot} = \downarrow 7,2,6,0,7,8,2,6, = \downarrow 0,0,0,0,0,0,0,0,\ 69314718, = \downarrow \overset{8}{\ } 69314718.$$

The 8 being omitted, the expression is written

$$2 = \downarrow 69314718,$$

Then 69314718, is termed the dual logarithm of 2· and written

$$\downarrow,(2\cdot) = 69314718,$$

$$10\cdot = \overset{8}{\downarrow} 230258509,$$

$$\therefore \quad \downarrow,(10) = 230258509,$$

The diameter of the earth through the poles is said to be 41706091·152 feet, the dual logarithm of which is equal to the whole number 1615789463;

Then $\downarrow,(41706091\cdot 152) = 1615789463,$

DESCENDING BRANCH.

6. In this branch the arrow points vertically, and the comma is to the left of the digit and *above* it, while in the ascending branch the arrow points straight *down*, and the comma is to the right of the digit and *below* it.

One decimal in the base

$(\cdot 9)$ is represented by		$'1\uparrow$	in the 1st position.	
$(\cdot 9)^2$	,, ,,	$'2\uparrow$	,,	,,
$(\cdot 9)^3$	,, ,,	$'3\uparrow$	,,	,,
&c.		&c.		

Two decimals in the base

$(\cdot 99)$ is represented by		$'0'1\uparrow$ or $'1\underset{2}{\uparrow}$	in the 2d pos.	
$(\cdot 99)^2$	,, ,,	$'0'2\uparrow$ or $'2\underset{2}{\uparrow}$	,,	,,
$(\cdot 99)^3$	,, ,,	$'0'3\uparrow$ or $'3\underset{2}{\uparrow}$	,,	,,
&c.		&c.		

Three decimals in the base

$(\cdot 999)$ is represented by		$'0'0'1\uparrow$ or $'1\underset{3}{\uparrow}$	in the 3d p.	
$(\cdot 999)^2$	,, ,,	$'0'0'2\uparrow$ or $'2\underset{3}{\uparrow}$	,,	,,
$(\cdot 999)^3$	,, ,,	$'0'0'3\uparrow$ or $'3\underset{3}{\uparrow}$	,,	,,
&c.		&c.		

Hence, in both branches, if there be r decimals in any base, its powers, or dual digits, are placed in the rth position.

$$(\cdot 9)^3\,(\cdot 99)^2 \text{ is written } '3\,'2\uparrow$$
$$(\cdot 9)^7\,(\cdot 99)^5 \quad \text{,,} \quad \text{,,} \quad '7\,'5\uparrow$$
$$(\cdot 999)^3\,(\cdot 999999)^2\,(\cdot 99999999)^6 \text{ is written } '0\,'0\,'3\,'0\,'0\,'2\,'0\,'6\uparrow$$

A cipher being in the first and also in the second positions shows that no power of $\cdot 9$ or $\cdot 99$ is employed; the same may be said of other positions occupied by ciphers.

$$'0\,'0\,'3\,'0\,'0\,'2\,'0\,'6\uparrow \text{ may be written } '3\,'0\,'0\,'2\,'0\,'6\underset{3}{\uparrow}$$
$$= '3\underset{3}{\uparrow}\,'2\underset{6}{\uparrow}\,'6\underset{8}{\uparrow}$$

$$(\cdot 9)^3\,(\cdot 99)^2\,(\cdot 999)^5\,(9\cdot)\,(99\cdot)^2 \text{ is written } '3\,'2\,'5\uparrow\,'1\,'2$$

and may be put under the form

$$'4\,'4\,'5\uparrow(10^5) = '4\,'4\,'5\uparrow^5$$

The reduction of $'3\,'2\,'5\uparrow\,'1\,'2$ to $'4\,'4\,'5\uparrow^5$ is similar to that established for the ascending branch.

For

$$(1\cdot 1)^3\,(1\cdot 01)^2\,(1\cdot 001)^5\,(11\cdot)\,(101)^2 = 2,1,\downarrow 3,2,5, = {}^5\!\downarrow 4,4,5,$$

7. In the descending branch, as in the ascending, a dual number reduced to the eighth position is also called a dual logarithm, and must be considered negative, if the ascending dual logarithm is taken positive, and *vice versâ*. It will be shown hereafter, that

$$'1\uparrow = '10536052\underset{8}{\uparrow}$$
$$'0\,'1\uparrow = '1005034\underset{8}{\uparrow}$$
$$'0\,'0\,'1\uparrow = '100050\underset{8}{\uparrow}$$
$$'0\,'0\,'0\,'1\uparrow = '10000\underset{8}{\uparrow}$$

&c. &c.

Then

$$'2\,'3\,'4\,'5\,'6\,'7\,'8\,'9 \uparrow\; = \;'24544195 \underset{8}{\uparrow};$$

as in the ascending branch, the 8 designating the position is omitted in practice.

Again,

$$\cdot 765432110 = \;'3\,'0\,'1 \updownarrow 0,5,0,0,1,5,6,3,$$

It has been already shown that

$$\downarrow 0,5,0,0,1,5,6,3, \;=\; \overset{8}{\downarrow}\, 4976728,$$

and that

$$'3\,'0\,'1 \uparrow \;=\; '31708206 \underset{8}{\uparrow}$$

Then

$$\cdot 76543211' = \;'31708206 \overset{8}{\underset{8}{\updownarrow}} 4976728, = \;'26731478 \underset{8}{\uparrow}$$

$$\begin{array}{r} '31708206 \\ 4976728, \\ \hline '26731478 \end{array}$$

$\therefore$ The dual logarithm of the decimal ·76543211 is '26731478 written $\downarrow$, (·76543211) = '26731478 ;

These reductions are introduced to exemplify the notation. How to make all such reductions will be shown when the operations of the descending branch are being discussed.

8. How to find any *two* of the three corresponding numbers—

(NATURAL NUMBER); (DUAL NUMBER); (DUAL LOGARITHM);

by easy and direct processes, the remaining *one* being given.

Any dual logarithm may be compounded of multiples of 69314718(n) and 230258509(m), and a logarithm numerically less than (34657359) half the dual logarithm of 2.

If 230258509 alone is operated with, any logarithm may be compounded of multiples of 230258509, and a logarithm numerically less than (115129255) half the logarithm of 10.

$\frac{1}{2}n =$	34657359	$\frac{1}{2}m =$	115129255
$n =$	69314718	$m =$	230258509
$1\frac{1}{2}n =$	103972077	$1\frac{1}{2}m =$	345387764
$2n =$	138629436	$2m =$	460517019
$2\frac{1}{2}n =$	173286795	$2\frac{1}{2}m =$	575646274
$3n =$	207944154	$3m =$	690775528
	&c.		&c.

If the given logarithm be greater than $\frac{n}{2}$ by x, but less than n, then

$$n - \left(\frac{n}{2} + x\right) = \frac{n}{2} - x,$$

a logarithm less than $\frac{n}{2}$,

If the given logarithm be greater than n, but less than $1\frac{1}{2}n$ by y, then

$$\left(\frac{3n}{2} - y\right) - n = \frac{n}{2} - y$$

a logarithm less than $\frac{n}{2}$.

Again, if the logarithm be greater than $1\frac{1}{2}n$ by z, but less than $2n$, then,

$$2n - \left(\frac{3n}{2} + z\right) = \frac{n}{2} - z,$$

which is also less than $\frac{n}{2}$; and so on. A similar process of reasoning may be applied to

$$\tfrac{1}{2}m;\; m;\; 1\tfrac{1}{2}m;\; 2m;\; \&c.$$

9. Because

$$\downarrow 3{,}6{,}0{,}9{,}4{,}1{,}0{,}7{,} = \overset{8}{\downarrow} 34657359,$$

and

$$'3'3'0'3'4'1'0'1 \uparrow = '34657359 \underset{8}{\uparrow},$$

hence any dual logarithm may be reduced to a dual number whose first digit does not exceed 3, or '3; and by operating with the logarithms of $\downarrow 1,$; $\overset{2}{\downarrow} 1,$; $\overset{3}{\downarrow} 1,$; &c. and of $'1 \uparrow$; $'1 \underset{2}{\uparrow}$; $'1 \underset{3}{\uparrow}$; &c. in a manner similar to that explained with respect to 69314718, and 230258509 (8), succeeding digits after the first may be found so as not to exceed 5, or '5.

Dual logarithms of $\downarrow 1,$ and $'1 \uparrow$; $\overset{2}{\downarrow} 1,$ and $'1 \underset{2}{\uparrow}$; $\overset{3}{\downarrow} 1,$ and $'1 \underset{3}{\uparrow}$ may be arranged in the following order:

$$'1\,'\uparrow = '10536052 \qquad 1\,'\underset{2}{\uparrow} = '1005034 \qquad '1\,'\underset{3}{\uparrow} = '100050$$

$$\downarrow,1, = \;9531018, \qquad \overset{2}{\downarrow}\,1, = \;995033 \qquad \overset{3}{\downarrow},1, = \;99950,$$

A multiple of 10536052 may be involved so that the remainder will not exceed half of 10536052 = 5268026, which contains 1005034 five times but not six; the same may be said of half 1005034 = 502517, &c. and of half 9531018 = 4765509; &c. &c.

We have now arrived at these important conclusions, namely that with the dual logarithms of 10· and 2· ($\downarrow$,(10·) and $\downarrow$,(2·)) and their multiples together with a logarithm, numerically, not

greater than 34657359, or '34657359 the dual logarithms of all the natural numbers between

$$+\infty \text{ and } 0$$

are instantly determined.

The corresponding dual number may be put under the form (A),

$$'v_1\,'v_2\,'v_3\,'v_4\,'v_5\,'v_6\,'v_7\,'v_8 \;{}^{m}\!\updownarrow^{n}\; u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, \quad \text{(A)};$$

in which it is not necessary that either $'v$, or u_1, should exceed '3 or 3, and at least half these digits may be ciphers.

Therefore, to determine in a direct manner the natural number corresponding to a dual logarithm requires but little numerical labour, since (A) may assume the forms (B), (C), (D), &c.

$$'0\,'v_2\,'0\,'v_4\,'0\,'0\,'0\,'0 \;{}^{m}\!\updownarrow^{n}\; u_1, 0, u_3, 0, u_5, u_6, u_7, u_8, \quad \text{(B)}.$$

$$'v_1\,'0\,'0\,'0\,'v_5\,'v_6\,'v_7\,'v_8 \;{}^{m}\!\updownarrow^{n}\; 0, {}^{\backprime}u_2\, {}^{\backprime}u_3\, {}^{\backprime}u_4, 0, 0, 0, 0, \quad \text{(C)}.$$

$$'0\,'v_2\,'0\,'v_4\, 0\,0\,0\,0 \;{}^{m}\!\updownarrow^{n}\; u_1, 0, u_3, 0, u_5, u_6, u_7, u_8, \quad \text{(D)}.$$

&c.

In reducing a dual logarithm of the form B, C, D, &c. to a natural number, it is of no moment to have each of the last four digits not greater than 5, or '5 ; then, when the positions $u_5\, u_6\, u_7\, u_8$ are occupied, $v_5\, v_6\, v_7\, v_8$ become ciphers, and *vice versâ*.

Let $u_5\, u_6\, u_7\, u_8$ be a natural number composed of the dual digits u_5, u_6, u_7, u_8, and let $v_5\, v_6\, v_7\, v_8$ be the natural number composed of the dual digits $'v_5\,'v_6\,'v_7\,'v_8$; then

$$\overset{5}{\downarrow} u_5\, u_6\, u_7\, u_8 = 10000\, u_5\, u_6\, u_7\, u_8$$

and

$$'v_5\,'v_7\,'v_7\,'v_8 \underset{5}{\uparrow} = 100000000 - v_5\, v_6\, v_7\, v_8$$

These numbers being operated on by such dual numbers as

$$'0\,'v\,'0\,'0 \;\; m\updownarrow n \;\; u_1, 0, u_3, u_4,$$

$$'v_1, '0\,'0\,'0 \;\; m\updownarrow n \;\; 0, u_2, u_3, u_4,$$

$$'0\,'v_2\,'0\,'v_4 \;\; m\updownarrow n \;\; u_1, 0, u_3, 0,$$

&c.

the corresponding natural number will be produced.

10. The solution of the converse problem, that is, to find the dual logarithm of any given natural between

$$+\infty \text{ and } 0$$

requires no additional labour or skill, since any given number operated on by 10· may be found in one or other of the positions N_1 N_2 N_3 N_4 N_5 N_6 and 1·00000000 ; 2·00000000 and ·50000000 may be operated upon by a dual number whose first digit is not greater than $\downarrow$ 3, or '3 $\uparrow$ and also assume one of the positions N_1 N_2 N_3 &c.

1·00000000	1·00000000
N_1	N^4
1·41421356	·70710678
N_2	N_5
2·00000000	·50000000
N_3	N_6
3·16227766	·316227766

Reductions may be often simplified by multiplying or dividing numbers found near the positions N_1 N_2 N_3 &c. by 2.

A brief inspection and comparison of the numbers exhibited in the subjoined tabulated form will exemplify our first sketch of these important relations.

		Examples.
	$\boxed{100000000}$	
Numbers intervening.	$= \begin{cases} 11 = \downarrow 1, \\ 121 = \downarrow 2, \\ 1331 = \downarrow 3, \end{cases}$	1·32898724 = '0'0'2'0'1'2'3'3 $\updownarrow$ 3,0,0,5, 28441721,
	$\boxed{141421356}$	
Numbers intervening divided by 2.	$= \begin{cases} 729 = {}'3 \uparrow \\ 81 = {}'2 \uparrow \\ 9 = {}'1 \uparrow \end{cases}$	1670·74 = '2'0'0'0'1 $_8\updownarrow_1$ 0,3,1, '17988055
	$\boxed{200000000}$	
Numbers intervening divided by 2.	$= \begin{cases} 11 = \downarrow 1, \\ 121 = \downarrow 2, \\ 1331 = \downarrow 3, \end{cases}$	239·468438 = '0'1'0'5 $_2\updownarrow_1$ 2,0,0,0,3,4,2,6, 18010428,
	$\boxed{266000000}$	
Numbers intervening multiplied by 4.	$= \begin{cases} 11 = \downarrow 1, \\ 121 = \downarrow 2, \\ 1331 = \downarrow 3, \end{cases}$	2797·01465 = '0'0'3 $_4\updownarrow_{\bar{2}}$ 1,2,0,5,0,2,5,7, 11271191,
	$\boxed{345000000}$	
Numbers intervening multiplied by 2.	$= \begin{cases} 729 = {}'3 \uparrow \\ 81 = {}'2 \uparrow \\ 9 = {}'1 \uparrow \end{cases}$	3605· = '3'1'1'0'0'3'7'6 $_4\updownarrow_{\bar{1}}$ 0,0,0,0,2, '32711615
	$\boxed{499999999}$	
Numbers intervening multiplied by 2.	$= \begin{cases} 11 = \downarrow 1, \\ 121 = \downarrow 2, \\ 1331 = \downarrow 3, \end{cases}$	·054549625 = '0'1'0'2'0'1'0'0 $_{\bar{1}}\updownarrow_{\bar{1}}$ 1,0,2,0,3,0,0,0, 9733918,
	$\boxed{707106780}$	
Numbers intervening.	$= \begin{cases} 729 = {}'3 \uparrow \\ 81 = {}'2 \uparrow \\ 9 = {}'1 \uparrow \end{cases}$	819·672683 = '2'0'0'1 $_3\updownarrow$ 0,1,2,0,2,1,5,2, '18885019
	$\boxed{999999999}$	

11. From what we have stated, it is evident that if tables be employed, but two are required, one of the ascending branch ranging from

$$\downarrow 0,0,0,0,0,0,0,0, \text{ to } \downarrow 3,6,0,9,4,1,0,7,$$

and another of the descending branch ranging from

$$'0'0'0'0'0'0'0'0\uparrow \text{ to } '3'3'0'3'4'1'0'1\uparrow$$

with natural numbers and dual logarithms to correspond, proper reductions being made involving powers of both 10 and 2.

When powers of 10 only are involved, no reductions require to be made. Two tables, one of the ascending branch whose natural numbers range from

$$1\cdot00000000 \text{ to } 2\cdot9916113362; \qquad \text{(I)}$$

and another of the descending branch whose natural numbers range from

$$\cdot999999999 \text{ to } \cdot29916113362; \qquad \text{(II)}$$

are then required.

With such tables logarithmic operations may be effected by mere inspection, and natural numbers are prepared for logarithmic operations by simply changing the decimal point one, two, three, &c. decimal places to the right or left until the natural numbers are to be found between

$$1\cdot00000000\ldots \text{ and } 2\cdot99161113612\ldots$$

or between

$$\cdot299161136\ldots. \text{ and } \cdot99999999\ldots.$$

Since, to change the decimal point one, two, &c. places to the right or left being tantamount to multiplying or dividing by 10, 100, &c.; in resulting natural numbers, the position of the decimal point is more readily obtained than if the operation was performed by common logarithms.

It must be remembered that dual logarithms are whole numbers, those of the descending branch have a comma to the

left *above*, and those of the ascending to the right *below;* thus, the dual logarithm of 2· as well as the dual logarithm of ½ is the whole number 69314718 but written

$$\downarrow,(2\cdot) = 69314718,$$
$$\downarrow,(\cdot 5) = {}'69314718$$
$$\text{sum} = 00000000 = \downarrow,(1)$$

If the dual logarithms of the ascending branch be considered positive, those of the descending branch must be taken as negative, and *vice versâ.*

Although the calculations throughout this work are made without the use of tables, and by processes designedly rendered prolix for the sake of clearness, yet, before entering upon the general discussion of the descending branch, it may be necessary to show, when tables of logarithms are employed, how vastly superior tables of dual logarithms are to those of common logarithms in both accuracy and precision.

With what numbers must tables of dual logarithms (I) (II) (10) be entered to find the logarithms of

245·672	98·3657	4·846321
2·345678*	1·35672*	33·4455
·0012345	·4763	·8765432*
·001	·1	100·
1000·	9473·	1276·

The numbers marked with the * have the decimal point in the required position; to find the dual logarithms of the other numbers the tables (I) (II) must be entered with

2,	2·45672	(I).	·98365	(II).	·4846321	(II).
'3	1·2345	(I).	·4763	(II).	·334455	(II).
'3	1·	(I).	1·	(I).	1·	(I).
,3,	1·	(I).	·9473	(II).	1·276	(I).

2·45672 becomes the first number if the decimal point be

removed two places to the right, marked 2,; 1·2345 becomes the second number when the decimal point is removed three places to the left, marked '3; and so on.

Multiply 90·986868, 19·4334858, and 295·429627 continually together by dual logarithms.

$$
\begin{array}{lrl}
2, & \downarrow,(2·95429627) = & 108326047, \\
1, & \downarrow,(1·94334858) = & 66441255, \\
2, & \downarrow,(·90986868) = & '9445506 \\
\hline
1, & & 165321796, \\
\overline{6,} & 1, \quad \downarrow,(·1) = & '230258509 \\
\hline
 & \downarrow,(·52237607) = & '64936713 \text{ Result.}
\end{array}
$$

$\therefore$ Product = 522376·07

In practice, subtraction is avoided in all cases, by substituting the arithmetical complements of that class whose sum is numerically least for the logarithms.

It requires but a trifling inspection to decide which of the two class of logarithms has the greatest numerical value.

Work of the above Example in a practical form.

$$
\begin{array}{rl}
 & \bar{1}891673953 \text{ ar. co.} \\
 & \bar{1}33558745 \text{ ar. co.} \\
 & '9445506 \\
 & '230258501 \\
\hline
\downarrow,(·52237607) = & '64936713
\end{array}
$$

$\therefore$ Product = 522376·07

The result 0064936713 is written '64936713, since a whole number is not altered in value by prefixing ciphers to the left. Hence when dual logarithms are employed no allowance has

to be made on account of having employed arithmetical complements, which is one of the advantages of this over other systems.

The management of common logarithms is rendered difficult, because the decimal part is always taken positive, while the whole numbers or indices may be either positive or negative. Thus, the common logarithm of ·00012345 is made up of two parts, -4, and $+$ ·0914911, written $\bar{4}$·0914911.

Find the value of ·00285095 × 82·550825 × ·0092730306 by dual logarithms.

$$
\begin{array}{lll}
'3 & \downarrow,(2\cdot85095) = & = 104765225, \\
'2 & \downarrow,(\cdot92730306)\ \text{ar. co.} & = \bar{1}2452512 \\
\frac{2,}{'3} & \downarrow,(\cdot82550825)\ \text{ar. co.} & = \bar{1}80824393 \\
\hline
 & \downarrow,(2\cdot18239151) = & 78042130,
\end{array}
$$

$\therefore$ ·0021823915 1 = the required value.

12. To reduce a dual number of the ascending branch to a dual logarithm.

RULE.

To the dual number taken as a common number, add 31018 times the first digit, and 33 times the second digit; then subtract 5 times the first three digits, a cipher being inserted after each, from the sum and the remainder is the dual logarithm.

Demonstration.

Let $\downarrow u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8$ be the dual number to be reduced to a dual logarithm;

then, "Dual Arithmetic, a New Art," p. 212.

$$
\begin{aligned}
\downarrow, 1, &= 9531018, = 10000000 - 500000 + 31018 \\
\downarrow,^2 1, &= 995034, = 1000000 - 5000 + 33 \\
\downarrow,^3 1, &= 99950, = 100000 - 50 \\
\downarrow,^4 1, &= 10000 \\
\downarrow,^5 1, &= 1000 \\
\downarrow,^6 1, &= 100 \\
\downarrow,^7 1, &= 10 \\
\downarrow,^8 1, &= 1
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad \downarrow, u_1, &= 10000000u_1 - 500000u_1 + 31018u_1 \\
\downarrow, u_2, &= 1000000u_2 - 5000u_2 + 32u_2 \\
\downarrow, u_3, &= 100000u_3 - 50u_3 \\
\downarrow, u_4, &= 10000u_4 \\
\downarrow, u_5, &= 1000u_5 \\
\downarrow, u_6, &= 100u_6 \\
\downarrow, u_7, &= 10u_7 \\
\downarrow, u_8, &= 1u_8
\end{aligned}
$$

But $\quad \downarrow, u_1, + \downarrow, u_2, + \downarrow, u_3, + \&c. = \downarrow, u_1, u_2, u_3, \&c.$

also $\quad 10000000u_1 + 1000000u_2 + 100000u_3 + \&c.$

being tantamount to writing the dual number as a natural one; and $500000u_1 + 5000u_2 + 50u_3 = (100000u_1 + 1000u_2 + 10u_3)5$, which is the same as to say, five times the first, second, and third digits supposing a cipher placed after each; hence, the truth of the rule is established.

When the analysis of the ascending branch of "Dual Arithmetic, a New Art," was being drawn up, the Author first gave this *Rule*, with other short methods of reduction, and some peculiar examples to show, among other things, that when the Theorems of Taylor, Maclaurin (or rather of Stirling), Lagrange and Laplace failed to apply, the dual method was applicable in all cases without fault or failure.

Examples.

Ex. 1. Reduce $\downarrow 3,4,5,6,7,8,9,2,$ to a dual logarithm.

$$
\begin{array}{rl}
\downarrow 3,4,5,6,7,8,9,2, & \\
93054 & = 3 \times 31018 \\
132 & = 4 \times 33 \\
\hline
34661078, & \\
152025 & = 304050 \times 5 \\
\hline
\text{Dual logarithm} = 33140828, &
\end{array}
$$

Ex. 2. Reduce $\downarrow 7,2,6,0,7,8,2,6,$ to a dual logarithm.

$$
\begin{array}{rl}
2^{\cdot} = \downarrow 7,2,6,0,7,8,2,6, & \\
217126 & = 7 \times 31018 \\
66 & = 2 \times 33 \\
\hline
72825018 & \\
351030 & = 702060 \times 5 \\
\hline
\downarrow,(2^{\cdot}) = 69314718, &
\end{array}
$$

13. To reduce a dual logarithm of the ascending branch to a dual number.

When the given logarithm is greater than $\downarrow,(2^{\cdot})$, or $\downarrow,(10^{\cdot})$, we have shown (8), how it may be compounded of multiples of $69314718, = \downarrow,(2^{\cdot})$, and $230258509 = \downarrow,(10^{\cdot})$ and a logarithm numerically not greater than (34657359), half the logarithm of $2^{\cdot}$. Let the remainder thus found be of the ascending branch, and if it does not consist of eight places of figures, establish eight places by prefixing ciphers to the left; then apply the following

RULE.

Add once, twice, three times, &c. 500000 according as the first figure on the left of the sum, becomes respectively 1, 2, 3, &c.

Subtract 31018 times the first figure, which must not alter after the operation, but reappear in the remainder. Then add once, twice, three times, &c. 5000 according as the second figure to the left of the sum becomes respectively, 1, 2, 3, &c.; subtract 33 times the second figure, which must not change in the operation, but reappear in the remainder. Again, add once, twice, three times, &c. 50, according as the third figure of the sum becomes respectively 1, 2, 3, &c. and the dual logarithm is reduced to a dual number of eight digits.

This Rule is the converse of the last, (12) and requires no demonstration.

Examples.

Ex. 1. Reduce the dual logarithm 547842164, to a dual number.

If twice 230258509, and once 69314718, be taken from the given logarithms, the remainder will be 18010428,

$$
\begin{array}{rl}
18010428, & \\
500000 & = 1 \times 500000 \\
\hline
18510428 & \\
31018 & = 1 \times 31018 \\
\hline
18479410 & \\
40000 & = 8 \times 5000 \\
\hline
18519410 & \\
264 & = 8 \times 33 \\
\hline
18519146 & \\
250 & = 5 \times 50 \\
\hline
\downarrow 1,8,5,1,9,3,9,6, &
\end{array}
$$

$$\therefore\ {}^{2}\!\downarrow^{1},\ 1,8,5,1,9,3,9,6, = 547842164,$$

Ex. 2. Reduce the dual logarithm 211373490, to a dual number.

The given logarithm and '230258509 together gives '18485019 a logarithm of the descending branch, which may be reduced by Rule (15), which will be found on page 21.

Ex. 3. Reduce 69314718, to a dual number.

$$
\begin{array}{rl}
69314718, & \\
3500000 & = 7 \times 500000 \\
\hline
72814718 & \\
217126 & = 7 \times 31018 \\
\hline
72597592 & \\
10000 & = 2 \times 5000 \\
\hline
72607592 & \\
66 & = 2 \times 33 \\
\hline
72607526 & \\
300 & = 6 \times 50 \\
\hline
\end{array}
$$

Dual number = ↓ 7,2,6,0,7,8,2,6,

14. To reduce a dual number of the descending branch to a dual logarithm.

RULE.

Add to the dual number written as a natural number, five times the first three digits, supposing a cipher placed after each, 36052, multiplied by the first digit, and 34 multiplied by the second, the sum will be the dual logarithm.

Demonstration.

Since

$$\begin{aligned}
\cdot 9 \downarrow 1,1,0,1,0,0,0,1, &= 1 \\
\cdot 99 \downarrow 0,1,0,1,0,0,0,1, &= 1 \\
\cdot 999 \downarrow 0,0,1,0,0,1,0,0, &= 1 \\
\cdot 9999 \downarrow 0,0,0,1,0,0,0,1, &= 1 \\
\cdot 99999 \downarrow 0,0,0,0,1,0,0,0, &= 1, \\
\&c. \qquad &= \&c.
\end{aligned}$$

$$\therefore \quad \downarrow,(\cdot 9) + \downarrow, 1,1,0,1,0,0,0,1, = 0;$$

but $\downarrow, 1,1,0,1,0,0,0,1, = 10536052,$

$$\therefore \quad \downarrow,(\cdot 9) + 10536052, = 0.$$

In a similar way it may be shown that

$$\begin{aligned}
\downarrow,(\cdot 99) + 1005034, &= 0 \\
\downarrow,(\cdot 999) + 100050, &= 0 \\
\downarrow,(\cdot 9999) + 10000, &= 0 \\
\&c. \qquad &\&c.
\end{aligned}$$

$\therefore$ The dual logarithm of

$\cdot 9$ is negative, and represented by the number

10536052 written '10536052;

but $\cdot 9$ is written '1 $\uparrow$

and $\downarrow,(\cdot 9)$ is written '1 '$\uparrow$

In the same way it may be shown that the dual logarithm of '0 '1 $\uparrow$ is equal '1005034 written

$$'0\,'1\,'\!\uparrow \text{ or } '1\,'\!\uparrow_2 = '1005034$$

$$'0\,'0\,'1\,'\!\uparrow \text{ or } '1\,'\!\uparrow_3 = '100050$$

$$'0\,'0\,'0\,'1\,'\!\uparrow \text{ or } '1\,'\!\uparrow_4 = '10000$$

&c. &c.

$$\begin{array}{lllll} \therefore & 'v_1 '\uparrow = & 10000000 v_1 + & 500000 v_1 + & 36052 v_1 \\ & 'v_2 '\uparrow = & 1000000 v_2 + & 5000 v_2 + & 34 v_2 \\ & 'v_3 '\uparrow = & 100000 v_3 + & 50 v_3 & \\ & 'v_4 '\uparrow = & 10000 v_4 & & \\ & \&c. & \&c. & & \end{array}$$

But

$$'v_1 '\uparrow + 'v_2 '\uparrow + 'v_3 '\uparrow + \&c. = 'v_1 'v_2 'v_3 \&c. '\uparrow ;$$

and

$$10000000 v_1 + 1000000_2 + 100000 v_3 + \&c.$$

being tantamount to writing the dual number as a natural one, while

$$5(100000 v_1 + 1000 v_2 + 10 v_3) = 500000 v_1 + 5000 v_2 + 50 v_3 ;$$

which is the same as tying five times the first, second, and third digits supposing a cipher placed after each. Hence the truth of the rule is established.

Let it be required to reduce '6'6'0'6'8'2'0'2 $\uparrow$ to a dual logarithm.

$$\begin{array}{rl} '6'6'0'6'8'2'0'2 \uparrow & \\ 3030000 & = 5 \times 606000 \\ 216312 & = 6 \times 36052 \\ 204 & = 6 \times 34 \\ \hline '69314718 & \end{array}$$

15. To reduce a dual logarithm of the descending branch to a dual number.

Rule.

Subtract once, twice, three times, &c. 536052, according as the first figure on the left becomes 1, 2, 3, &c. which first figure must not alter but reappear in the remainder. Then subtract, once, twice, three times, &c. 5034, according as the second to

the left of the remainder become respectively 1, 2, 3, &c. Again, subtract once, twice, three times, &c. 50, according as the third figure of the remainder becomes 1, 2, 3, &c. respectively. Thus the dual logarithm is reduced to a dual number of eight descending dual digits. This rule being the converse of the last requires no demonstration.

It is not necessary to say more in this place respecting the descending branch as it will be fully discussed hereafter, both independently and in conjunction with the ascending branch. The practical calculator, however, who requires to be independent of tables and possess means by which the accuracy of his results may be readily tested, can see, we have no doubt, from these preliminary propositions and examples, how dual arithmetic completely and simply furnishes these requirements. At the same time an operator employing tables will easily perceive how incomparably superior tables of dual logarithms are to those of common logarithms.

CHAPTER II.

RESULTS OBTAINED, BY DUAL DEVELOPMENTS, OF SIMPLE OPERATIONS CONVENTIONALLY EXPRESSED IN ALGEBRAIC LANGUAGE. PRODUCTS, QUOTIENTS, POWERS, AND ROOTS.

Recapitulation of Conventional Arrangements and Notation.

16. THE dual logarithm of A is written $\downarrow,(\mathrm{A})$.

If $$\mathrm{U} = \downarrow u_1,\ u_2,\ u_3, \ldots\ldots = \overset{s}{\downarrow} u,$$

then $$\downarrow,(\mathrm{U}) = u, \quad \text{and} \quad \downarrow, u_1,\ u_2,\ u_3, \ldots\ldots = u,$$

$$\downarrow,(2) = 69314718, \qquad \downarrow,(10) = 230258509,$$

A comma is placed on the right of positive dual logarithms, and on the left of negative dual logarithms. Thus, 34567844, is a positive dual logarithm, the same as + 34567844 a positive whole number; and '45678921 is a negative dual logarithm, the same as − 45678921 a negative whole number. A dual logarithm is changed from positive to negative, and from negative to positive, by simply changing the position of the comma from right to left or from left to right, as the case may be. The arithmetical complements of dual logarithms (ar. co.) do not retain the comma. *To find the arithmetical complement of a dual logarithm;*—begin at the left, set down minus 1, written $\bar{1}$, then take each of the figures from 9 except the last figure on the right, which must be taken from 10.

45665423, dual log.	'76543298 dual log.
$\bar{1}54334577$ ar. co.	$\bar{1}23456702$ ar. co.

Logarithms of the ascending branch have the comma to the right, while the comma is to the left of logarithms of the descending branch.

Examples.

Ex. 1. Find the cube root of

$$\frac{(1865·655)(·02691098)(848·21877)}{(328·7077)(·134659655)(6296·168)}$$

Ans. ·53462388

Representing by (A) and (B) the numerator and denominator of the fraction.

(A)			(B)		
3,	↓,(1·865655)	= 62361219,	3,	↓,(·3287077)	= '111258635
'2	↓,(2·691098)	= 98994929,	'1	↓,(1·34659555)	= 29758037,
3,	↓,(·84821877)	= '16461667	4,	↓,(·6296168)	= '46264392
4,		144894481,	6,		'127764990
'6		127764990,			
3)'2		272659471,			
0 and '2 over		'460517018	↓,(10^2)		
For the cube root divide by		3)'187857547			
'5'9'8'9'3'2'1'6↑ = ·53462388		'62619182	½ ↓ 0,6,7,2,5,8,8,8, = ·53462388		

These dual logarithms and the natural number answering to the resulting dual logarithm ('62619182), may be independently calculated at once by the methods and rules laid down in the preceding chapter, or by any of those detailed in "Dual Arithmetic, a New Art."

17. If tables of dual logarithms be employed, like those described (11) ranging from 1·00000000 to 2·99161136 and from ·299161136 to ·999999999, the required numbers are obtained by mere inspection, and with far less inconvenience than with a table of common logarithms.

The number of places of figures the decimal point has to be removed to the right or left, *being noted,* (C), the dual logarithm of any number, as 1·865655 may be employed to represent the logarithm of

·001865655	'3	
·01865655	'2	
·1865655	'1	
1·865655	0	(C)
18·65655	1,	
186·5655	2,	
1865·655	3,	
&c.		

In practice it is not necessary to set down, as at (A), (B), (C), the number of places which the decimal point is removed to the right or left to produce the number to be operated with. Indeed, the final number ('2) found by taking the amount of B (6,) from the amount of A (4,) may be instantly counted before commencing to operate.

Referring to (A), (3,) with the comma to the right is considered positive, and signifies that the period is to be removed three places to the right to bring 1·865655 to 1865·655. Again '2 with the comma to the left, is considered negative, and indicates that the period removed two places to the left will bring 2·691098 to ·02691098; and so on.

If the value of

$$\sqrt[3]{\frac{(1865\cdot655)(2691\cdot098)(848\cdot21877)}{(328\cdot7077)(1\cdot34659655)(629\cdot6168)}}$$

had to be found, then (A) and (B) would become

(A)	(B)		
3,	3,		9,
3,	0		'6
3,	3,	For the cube root divide by 3)	3,
9,	6,		1, and 0 over.

In the first case '2 does not contain 3; then '2 is over and twice the dual logarithm of 10 is incorporated under a negative form with the amount 272659471,; in the latter case, the natural number corresponding to the dual logarithm answering to the amount 272659471, has to be multiplied by 10; or the natural number answering to 42400962, must be multiplied by 10.

For,

	272659471,	
	230258509,	↓, (10)
↓, (1·52807622) =	42400962,	

∴ 152·807622 is the result in the latter case.

It is almost unnecessary to remark, that, instead of adding and subtracting as above, the resulting logarithm may be found by addition.

18. Before taking the arithmetical complements, the commas of the logarithms of the dividing factors have to be changed from right to left, or from left to right, as the case may be. Then, the arithmetical complements of the logarithms with the comma to the left, if that class be *least* in amount, have to be taken. On the contrary, the arithmetical complements of the other class, if that class be *less* in amount, are to be taken, (12).

	↓, (1·865655)	......	$\bar{1}37638781$	ar. co.
	↓, (2·691098)	......	$\bar{1}01005071$	ar. co.
	↓, (·84821877)	......	'16461667	
Divisor	↓, (·3287077)	......	$\bar{1}888741365$	ar. co.
Divisor	↓, (1·34659655)	......	'29758937	
Divisor	↓, (·6296168)	......	$\bar{1}53735608$	ar. co.
	↓, (10^2)	......	'460517018	
			3) '187857547	
	↓, (·53462388)	......	'62619182	

It is evident that no allowance has to be made on account of having to employ arithmetical complements, which is one of the

many advantages of this over any other system. The management of common logarithms is rendered difficult because the decimal part is always taken as positive, and is the only part given in tables, while the whole number or indices may be either positive or negative; for example, the common logarithm of .00012345 is made up of two parts, -4 and $+9{\cdot}014911$, written $\bar{4}{\cdot}0914911$.

Ex. 2. *Find the cube root of* $\dfrac{34{\cdot}5 \times 76{\cdot}3 \times 355{\cdot}}{84{\cdot}0 \times 36{\cdot}6 \times 887{\cdot}}$, *by the ascending branch involving powers of* 10· *and* 2· *and also by the shorter method by addition and powers of only* 10.

Since

$$\frac{3{\cdot}45 \times 7{\cdot}63 \times 3{\cdot}55}{8{\cdot}40 \times 3{\cdot}66 \times 8{\cdot}87} = \frac{34{\cdot}5 \times 76{\cdot}3 \times 355{\cdot}}{84{\cdot}0 \times 36{\cdot}6 \times 887{\cdot}},$$

then

$$\begin{array}{ll} \downarrow,(3{\cdot}45), = 123837420, & \downarrow,(8{\cdot}40) = 212823170, \\ \downarrow,(7{\cdot}63), = 203208780, & \downarrow,(3{\cdot}66) = 129746310, \\ \downarrow,(3{\cdot}55), = \underline{126694750,} & \downarrow,(8{\cdot}87) = \underline{218267470,} \\ \qquad\qquad 453740950, & \qquad\qquad 560836950, \end{array}$$

These dual logarithms may be found at once by the methods given in the previous chapter, and in "Dual Arithmetic, a new Art," page 213. These methods, however, will be considerably simplified in the course of the present work.

$$\begin{array}{rl} & +453740950, = \downarrow,(3{\cdot}45) + \downarrow,(7{\cdot}63) + \downarrow,(3{\cdot}55) \\ & -\underline{560836950,} = \downarrow,(8{\cdot}40) + \downarrow,(3{\cdot}66) + \downarrow,(8{\cdot}87) \\ \text{For the cube root divide by } 3) & -\underline{107096000,} \\ & -\ 35698667, = \text{log of required cube root.} \\ \downarrow,(2) & \underline{\ 69314718,} \\ & \ 33616051, \text{ log of } 1{\cdot}39956362 \text{ the half} \end{array}$$

of which is ·69978181 the cube root required. The nearest number corresponding to the dual log $\div$ 35698667, written

'35698667 may be found by the methods indicated above. If required, the number could as easily be obtained to eight, nine, &c. places of decimals. The result − 2365333, being negative, the addition of log 2 renders it positive, then 1·39956362 has to be divided by 2.

Second and shorter method.

From	2,		$\downarrow,(\cdot345) =$	'106421089
	3,		$\downarrow,(\cdot763) =$	'27049728
	3,		$\downarrow,(\cdot355) =$	'103563757
Take	2,	Divisor.	$\downarrow,(\cdot840)$	$\bar{1}82564661$ ar. co.
	3,		$\downarrow,(\cdot366)$	$\bar{1}899487802$ ar. co.
	3,		$\downarrow,(\cdot887)$	$\bar{1}88008963$ ar. co.
diff.	0		3)	'107096000
			$\downarrow,(\cdot69978181) =$	'35698667 = '3'4'0'7'0'3'7'5 ↑

It requires more skilled labour to obtain similar results by common logarithms, for common logs are made up of a combination of whole numbers and decimals; besides, as before observed, the common log of a decimal fraction is part positive and part negative, while dual logarithms are always whole numbers, either positive or negative.

Ex. 3. *Find the value of* $\left\{\frac{\cdot\sqrt[3]{0376 \times 19\cdot8}}{\sqrt{66\cdot8 \times \cdot947}}\right\}^{\frac{3}{7}}$ *to nine places of decimals.*

$$66\cdot8 \times \cdot947 = 6\cdot68 \times 9\cdot47$$

$$\cdot0376 \times 19\cdot8 = \cdot376 \times 1\cdot98 = (\tfrac{1}{10})\ 3\cdot76 \times 1\cdot98$$

$\downarrow,(1\cdot98)$	68309680,	$\downarrow,(6\cdot68) =$	189911790,
$\downarrow,(3\cdot76)$	132441890,	$\downarrow,(9\cdot47) =$	224812880,
	+ 200751570,	Square root 2)	414724670,
$\downarrow,(10)$	− 230258509,		207362335,
For cube root 3)	− 29506939,		
	− 9835646,		

$$
\begin{array}{rl}
& -\ 9835646, \\
& \underline{-\ 207362335,} \\
& -\ 217197981, \\
& \underline{\qquad\qquad\quad 2} \\
7) & \underline{-\ 434395962,} \\
& -\ 62056566, \\
\downarrow,(2) = & \underline{+\ 69314718,} \\
& \quad 7258152, = \downarrow,(1{\cdot}07528041) \\
2) & \underline{1{\cdot}07528041} \\
& {\cdot}537640205 \text{ required root.}
\end{array}
$$

Shorter method involving ↓,(10.) *but not* ↓,(2.)

(A)

$$
\begin{array}{llr}
1, & \downarrow,(1{\cdot}98) & \bar{1}31690320 \text{ ar. co.} \\
'1 & \downarrow,({\cdot}376) = & \underline{'97816619} \\
\bar{0} & \qquad 3) & \underline{'29506939} \\
& & '9835646 \\
& & \bar{1}77103826 \\
& & \underline{'230258509} \\
& & '217197981 \\
& & \underline{2} \\
& 7) & \underline{'434395962} \\
& & '62056566 \quad = \downarrow,({\cdot}537640205)
\end{array}
$$

(B)

$$
\begin{array}{llr}
2, & \downarrow,(.668) = & '40346718 \\
\underline{0} & \downarrow,({\cdot}947) = & \underline{'5445629} \\
2)\ \bar{2}, & 2) & \underline{'45792347} \\
1, & & '22896174 \\
& \text{ar. co.} & \bar{1}77103826
\end{array}
$$

19. The small numbers under (A) and (B) may be omitted in practice, and also ar. co. to designate the arithmetical complement, which is sufficiently indicated by being always preceded by minus one ($\bar{1}$).

Ex. 4. *Multiply* 5486·48128 *by* 386·344448 *and take the square root of the product; and also the fifth root.*

$$
\begin{aligned}
5486{\cdot}48128 &= 2^2 \,.\, 10^3 \,.\, (1{\cdot}37162032) \\
386{\cdot}344448 &= 2 \ \,.\, 10^2 \,.\, (1{\cdot}93172224)
\end{aligned}
$$

↓,(1·37162032) = 31599277,
↓,(1·93172224) = 65841199,
5 ↓,(10) = 1151292545,
3 ↓,(2) = 207944154,

2) 1456677175,

728338588,
↓,(10³) = 690775527,

37563061, log of 1·45590923.

∴ 1455·90923 is the required square root.

Again,

5) 1456677175,

291335435,
↓,(10) = 230258509,

61076926, = (1·84184672)

∴ 18·4184672 is the required fifth root.

Practical Method.

4, ↓,(·548648128) $\bar{1}39970204$
3, ↓,(·386344448) $\bar{1}04897408$

2)$\bar{7}$,

$\bar{3}$, and 1, over ↓,(10.) = 230258509,

2) 75126121,

↓,(1·45590923) = 37563061,

∴ 1455·90923 = square root.

$\bar{1}39970208$
5)$\bar{7}$, $\bar{1}04897408$
$\bar{1}$, and 2, over ↓,(10²) = 460517018,

5)305384630

↓,(1·84184672) = 61076926,

∴ 18·4184672 = fifth root.

20. It may be contended by some that such results as we have obtained might be more conveniently and expeditiously found by a common table of logarithms. To which we reply that without the use of tables of dual logarithms our methods might require more labour, yet their results may be depended upon and tested for their accuracy up to the last figure. It must be remembered that the effective use of a table of common logarithms is acquired only from considerable practice, and even when this skill is attained, we have no means of testing our results. All that we can be sure about, supposing the tables to be correct, is that the first five or six digits of the products, powers or roots of numbers, obtained from logarithmic tables to seven places of decimals can be depended on, though we have no independent methods of testing the results.

In every respect tables of dual logarithms such as described (I. and II.) in section (11) page 12, are incomparably superior to any tables of logarithms that have hitherto been calculated.

Ex. 5. *Find the* $\frac{3}{5}$ *root of* ·832516529.

$$\cdot 832516529 = \tfrac{8}{10}(1\cdot 04064566) = \tfrac{8}{10}\downarrow 0,4,0,0,4,0,0,3,$$

$$\begin{array}{lrr}
\downarrow 0,4,0,0,4,0,0,3, & = & 3984135 \\
\downarrow,(8) & = & \underline{207944154} \\
 & + & 211928289 \\
\downarrow,(10) & - & \underline{230258509} \\
 & - & 18330220 \\
 & & \underline{3} \\
 & 5)- & \underline{54990660} \\
 & - & 10998132 \\
\downarrow,(2) & + & \underline{69314718} \\
 & & 58316586 = \downarrow 6,1,1,3,5,4,9,5, = 1\cdot 79170165
\end{array}$$

$$\begin{array}{r}
2)\underline{1\cdot 79170165} \\
\cdot 895850825 \text{ the required root.}
\end{array}$$

Shorter Method.

$$\downarrow,({\cdot}832516529 = {'}18330220$$
$$3$$
$$5){'}54990660$$
$$\downarrow,({\cdot}895850825 = {'}10998132$$

Before the introduction of Dual Arithmetic, the calculator would find it difficult to solve such a question as the last without tables. By means of a table of common logarithms, he might worm out ·89585, and no more. But whether this result was right or wrong, he could not decide with any certainty, unless he constructed a whole table of logarithms.

In many cases tables of common logarithms are found to be very inconvenient. Thus, if it were required to find the value of $({\cdot}07)^{\cdot 07}$ by logarithms, the operation would be as follows,

$$\text{common log of } ({\cdot}07) \ldots\ldots - 2 + {\cdot}8540980$$
$${\cdot}07$$
$$- {\cdot}14 + {\cdot}059156860 = - {\cdot}080843 14;$$

a result which is negative, and therefore cannot be found in a table of common logarithms.

But,

$$- {\cdot}08084314 = - 1(1 - {\cdot}08084314) = - 1 + {\cdot}91915686,$$

written $\bar{1}{\cdot}91915686$

REDUCTION OF ORDINARY FORMULÆ FREQUENTLY EMPLOYED.

21. Let $2^p\ 10^q \downarrow u_1,\ u_2,\ u_3, \ldots\ldots$ be the dual number corresponding to the common number N, and n the dual logarithm of $\downarrow u_1, u_2, u_3, \ldots\ldots$ that is $\downarrow u_1,\ u_2,\ u_3, \ldots\ldots = \overset{8}{\downarrow} n,$

Then

$$N = 2^p\ 10^q \downarrow u_1,\ u_2,\ u_3, \ldots\ldots = 2^p\ 10^q \downarrow \overset{8}{n},;$$

To avoid the use of the bases 2 and 10 in the expression $2^p\ 10^q$ we may write it under the form $p'\ q$, the inverted comma between the p and q indicating that the base 2 and 10 are suppressed.

Thus,

$$N = p'\ q \downarrow u \ldots = p'\ q \downarrow n,$$

In applying the same notation to any other common number M, then,

$$M = r'\ s \downarrow u_1 \ldots = r'\ s \downarrow m, \qquad \downarrow,(M) = r \downarrow,(2) + s \downarrow,(10) + m;$$

$$\therefore\ \sqrt{N^2 + M^2} = \{2p'\ 2q \downarrow 2n, + 2r'\ 2s \downarrow 2m,\}^{\frac{1}{2}}$$

Of m and n let m be the greater,

Then,

$$(N^2 + M^2)^{\frac{1}{2}} = r'\ s \downarrow n, \left\{ \frac{2p'\ 2q}{2r'\ 2s} + \frac{\downarrow 2m}{\downarrow 2n} \right\}^{\frac{1}{2}}$$

$$= r'\ s \downarrow n\ \{2(p - r)'\ 2(q - s) + \downarrow 2(m - n),\}^{\frac{1}{2}}$$

$\{2(p - r)'\ 2(q - s) + \downarrow 2(m - n),\}$ may always be reduced to the form $2\,a'\ 2\,b \downarrow 2c$, and $\downarrow 2(m - n)$ is less than 2, since neither m nor n can ever be equal to 2. The square root of $2\,a'\ 2\,b \downarrow 2\,c$, $= a'\ b \downarrow c$,

$$\therefore\ (N^2 + M^2)^{\frac{1}{2}} = (a + r)'(b + s) \downarrow n, \downarrow c, = (a + r)'\ (b + s) \downarrow (n + c),$$

Ex. 6. Find the value of

$$\sqrt{(635 \cdot 297388)^2 + (2536 \cdot 92174)^2}$$

Here

$$M = 635 \cdot 297388 = 2^2\ 10^2\ (1 \cdot 58824347) = 2'\ 2 \downarrow 46262870,$$
$$= r'\ s \downarrow m,$$

$$N = 2536 \cdot 92174 = 2^1\ 10^3\ (1 \cdot 26846087) = 1'\ 3 \downarrow 23780432,$$
$$= p'\ q \downarrow n,$$

The calculations with 2 and 10 are extremely simple.

$$\downarrow 2\,(m - n), = \downarrow 44964876, = 1{\cdot}56776145$$

$$\frac{2^2\ 10^6}{2^4\ 10^4} = \frac{10^2}{2^2} = \frac{2\,p^{\text{‘}}\ 2\,q}{2\,r^{\text{‘}}\ 2\,s} = 2\,(p - r)^{\text{‘}}\ 2\,(q - s)$$

$$\frac{10^2}{2^2} + 1{\cdot}56776145 = \frac{10^2}{2^2}\,(1 + {\cdot}062710458) = \frac{10^2}{2^2} \downarrow 6082269, \text{ and}$$

corresponds to the expression $2\,a^{\text{‘}}\ 2\,b \downarrow 2\,c$,

Hence,

$$\sqrt{(635{\cdot}297388)^2 + (2536{\cdot}92174)^2}$$

$$= 2^2\,10^2 \downarrow 23780432,\ \frac{10}{2} \downarrow 3041135$$

$$= 2 \times 10^3 \downarrow 26821567, = 2615{\cdot}2587$$

And in this way the value of any expression of the form $\sqrt{M^2 + N^2}$ can be obtained, whatever numbers M and N may be.

It may here be noted that such an expression as this cannot be solved by common logarithms, except by adapting the expression $\sqrt{M^2 + N^2}$ to logarithmic computation by means of the introduction of a subsidiary angle—a method which requires the use of logarithmic tables of Trigonometrical functions.

The value of $(N^s \pm M^s)^{\frac{q}{s}}$, may be found in a similar manner.

Thus to find the value of $\sqrt[3]{(4567{\cdot}8346)^{\frac{2}{3}} - (4321{\cdot}695)^{\frac{2}{3}}}$ to nine places of figures.

$$(4567{\cdot}8346)^{\frac{2}{3}} = \{2^2\,10^3(1{\cdot}14195865)\}^{\frac{2}{3}} = \{2^2\,10^3 \downarrow 13274495,\}^{\frac{2}{3}} = 2^{\frac{4}{3}}\,10^2 \downarrow 88496$$

$$(4321{\cdot}695)^{\frac{2}{3}} = \{2^2\,10^3(1{\cdot}08042375)\}^{\frac{2}{3}} = \{2^2\,10^3 \downarrow 7735337,\}^{\frac{2}{3}} = 2^{\frac{4}{3}}\,10^2 \downarrow 51568$$

$$36927$$

$\frac{1}{3}$ of 5156891 = 1718964 and $\downarrow 3692772, = 1·03761800$; then
$\{(4567·8346)^{3} - (4321·695)^{3}\}^{\frac{1}{3}} = 2^{\frac{1}{3}} 10^{3} \downarrow 1718964, \{1·03761800 - 1\}^{\frac{1}{3}}$

$$\left\{1·03761800 - 1\right\}^{\frac{1}{3}} = \left\{\frac{2}{10^{2}}, (1·88090000)\right\}^{\frac{1}{3}} = \left\{\frac{2}{10^{2}} \downarrow 63175040,\right\}^{\frac{1}{3}}$$

$$= \frac{2^{\frac{1}{3}}}{10^{\frac{2}{3}}} \downarrow 21058347,$$

$2^{\frac{1}{3}} \downarrow 1718964, \downarrow 21058347, = \downarrow 76688758, = 2·15305460$, the required root. The general reasoning employed in the last example may be applied to all examples of this class. The more complex methods of reduction are designedly employed.

To find the value of $\sqrt{M^{2} - N^{2}}$, which cannot be found by ordinary logarithms without the use of subsidiary angles by the dual method.

Ex. 7. To find the value of $\sqrt{(15676·06501)^{2} - (12649·47259)^{2}}$ to nine places of figures.

Since,

$$(A + B)(A - B) = A^{2} - B^{2}$$

Put

$$M = A + B, \text{ and } N = A - B;$$

According to the notation before employed,

$$M = p^{\prime}q \downarrow u, \ldots\ldots = p^{\prime}q \downarrow m,$$
$$N = r^{\prime}s \downarrow u, \ldots\ldots = r^{\prime}s \downarrow n,$$

$$\sqrt{A^{2} - B^{2}} = \sqrt{MN} = \frac{p + r}{2} \cdot \frac{q + s}{2} \downarrow \frac{m + n}{2},$$

$M = 28325·5376 = 1^{\prime}4 \downarrow 34803152,$
$N = 3026·59242 = 1^{\prime}2 \downarrow 41429021,$

$2^{\prime}7 \downarrow 76232173,$
$1^{\prime}\frac{7}{2} \downarrow 38116087, = 2^{1}\, 10^{3^{\prime}} \downarrow 38116087,$
$= 9259·04208.$

EXAMPLES ON THE NOTATION FOR THE BASES 2 AND 10, APPLIED TO QUESTIONS OF INTEREST AND ANNUITIES.

22. The notation previously employed will be rendered more complete, in many cases, by putting one letter instead of two to the left of the sign $\downarrow$, to express the combined powers of 2 and 10. It will be found that this contraction will neither render expressions obscure, nor curtail their generality.

$$N = 2^p\ 10^q \downarrow u_1, u_2, u_3, \ldots\ldots = 2^p\ 10^q \overset{8}{\downarrow} n,$$

was expressed under the form

$$N = p' q \downarrow u \quad \ldots\ldots = p' q \downarrow n,$$

which by the method above indicated will become

$$N = \mathrm{n} \downarrow u \ldots\ldots = \mathrm{n} \downarrow n, ;$$

where the italic (n) represents the dual logarithm of the dual number

$$\downarrow u \ldots\ldots \text{ or } \downarrow u_1, u_2, u_3, \ldots\ldots ;$$

and the roman (n) represents

$$p' q \text{ or } 2^p\ 10^q$$

$$\text{Log of N}, = \downarrow,(\mathrm{N}) = \downarrow,(\mathrm{n}) + n,$$

$$\{2^p\ 10^q\}^v = 2^{vp}\ 10^{vq} = vp' vq = v\mathrm{n},$$

v being a whole number or a fraction, positive or negative.

Again, let

$$M = 2^r\ 10^s \downarrow u_1, u_2, u_3, \ldots\ \ . = 2^r\ 10^s \overset{8}{\downarrow} m,$$

Or

$$M = r' s \downarrow u_1, \ldots\ldots = r' s \downarrow m,$$

Or

$$M = \mathrm{m} \downarrow u_1, \ldots\ldots = \mathrm{m} \downarrow m,$$

Then

$$\mathrm{M} \times \mathrm{N} = (\mathrm{m} + \mathrm{n}) \downarrow (u + u_1) \ldots\ldots = (\mathrm{m} + \mathrm{n}) \downarrow (m + n),$$

so that

$$\mathrm{m} + \mathrm{n} \text{ indicates } (p + r)^{\text{c}} (q + s), \text{ or } 2^{p+r}\, 10^{q+s}.$$

And

$$\frac{\mathrm{M}}{\mathrm{N}} = (\mathrm{m} - \mathrm{n}) \downarrow (u_1, - u) \ldots\ldots = (\mathrm{m} - \mathrm{n}) \downarrow (m - n_,)$$

Then

$$\mathrm{m} - \mathrm{n} \text{ indicates } (r - p)^{\text{c}} (s - q) \text{ or } 2^{r-p}\, 10^{s-q}$$

Hence, it must be observed, that in the addition and subtraction of the small roman letters to the left of $\downarrow$, the powers of 2 can only combine with the powers of 2; and powers of 10 with powers of 10.

For examples in interest and annuities:

Let

P denote the principal in pounds.

I the rate of interest.

i the interest of one pound for a year, $= \dfrac{\mathrm{I}}{100}$.

Y the time in years.

M the amount of P at compound interest.

$\mathrm{R} = (1 + i)$ the amount of one pound in a year.

23. Then it is shown by elementary writers that,

$$\mathrm{M} = \mathrm{P}\,\mathrm{R}^{\mathrm{Y}}.$$

Ex. 8. *How much would* £327·436 (P), *amount to in* (Y) 27 *years at* $3\frac{1}{2}$ (I), *per cent per annum, compound interest?*

Here

$$P = 327·436$$
$$Y = 27$$

And

$$I = 3\frac{1}{2}$$

And P, Y, and I are given to find M.

P being a common number can be expressed, as shown above, in the form

$$2^m\ 10^n \downarrow p, \text{ or } \mathrm{p} \downarrow p,$$

And R being less than 2, since

$$R = 1 + \frac{I}{100} = 1 + \frac{3·5}{100}$$

can be expressed in the form

$$R = \downarrow r,$$

$$\therefore \quad R^Y = \downarrow Yr,$$

$$\therefore \quad M = PR^Y = \mathrm{p} \downarrow p, \downarrow Yr, = \mathrm{p} \downarrow (Yr + p),$$

$$R = 1·035 = \downarrow 3440148,$$

$$R^Y = \downarrow 27(3440148), = \downarrow 92883996,$$

$$P = 2^1\ 10^2\ (1·63718000) = \mathrm{p} \downarrow p, = 2^1\ 10^2 \downarrow 49297588,$$

$$M = PR^Y = 2^1 10^2 \downarrow 49297588, \downarrow 92883996,$$

$$= 2^1 10^2 \downarrow 142181584, = 2^1\ 10^2 \downarrow \{2(69314718) + 3552148\}$$
$$= 2^3\ 10^2 \downarrow 355148, = £828·927884.$$

Ex. 9. *How much money must be placed out at compound interest to amount to £3,000 in 20 years, at 5 per cent.?*

24. Here M, R, and Y are given to find P.

$$P = \frac{M}{R^Y} = \frac{m \downarrow m,}{\downarrow Yr,} = m \downarrow (m - Yr),$$

But

$$M = 3000 = 2^1\, 10^3 \downarrow 40546562, = m \downarrow m,$$
$$Y = 20.$$
$$R = 1{\cdot}05 = \downarrow 4879021 = \downarrow r,$$

$$\therefore\quad R^Y = \downarrow Yr, = \downarrow (4879021 \times 20),$$
$$= \downarrow 97580420, = 2 \downarrow 28265702,$$

$$P = \frac{m \downarrow m,}{\downarrow Yr,} = \frac{2^1\, 10^3 \downarrow 40546562,}{2 \downarrow 28265702,}$$
$$= 10^3 \downarrow 12280860,$$
$$= £1130{\cdot}66794$$

Ex. 10. *At what interest must £422·3575 be placed out to amount to £666·666 in 15 years?*

25. Here M, P, and Y are given to find I.

$$R = \left(\frac{M}{P}\right)^{\frac{1}{Y}} = \frac{(m - p) \downarrow (m - p),}{Y} = \frac{m - p}{Y} \downarrow \frac{m - p}{Y},$$

$$M = 666{\cdot}666 = 2^2\, 10^2 (1{\cdot}6666500) = 2^2\, 10^2 \downarrow 51083464, = m \downarrow m,$$
$$P = 422{\cdot}3575 = 2^2\, 10^2 (1{\cdot}055893750) = 2^2\, 10^2 \downarrow 5438761, = p \downarrow p,$$
$$\frac{M}{P} = \downarrow 45644703,$$

$$\therefore\quad R = \downarrow \frac{m - p}{Y}, = \downarrow \frac{45644703}{15}, = \downarrow 3042980, = 1{\cdot}03089748.$$

$$R = (1 + i)$$

$$\therefore\quad i = {\cdot}03089748 \text{ and } \frac{I}{100} = i$$

$$\therefore\quad I = 3{\cdot}089748, \text{ the interest, or rate per cent.}$$

Ex 11. *What will £927 10s. amount to in* 18 *years, at* 6 *per cent. compound interest, payable half-yearly?*

26. Here

$$\text{R, Y, P, are given to find M.} \quad M = PR^{Y}$$

$$R = 1{\cdot}03;\ Y = 36;\ P = 927{\cdot}5.$$

$$R = \downarrow r, = \downarrow 2955883,$$

$$R^{Y} = \downarrow Yr, = \downarrow 36(2955883). = 2\downarrow 37097070,$$

$$P = p\downarrow p, = 2^{3}\ 10^{2}\downarrow 14788110,$$

$$M = PR^{Y} = P\downarrow(Yr + p), = 2^{4}\ 10^{2}\downarrow 51885180, = £2688{\cdot}159856.$$

Ex. 12. *In how many years will a sum of money, lent at* 5 *per cent. per annum, compound interest, double itself?*

27. Here M, P, and R are given to find Y.

$$M = 2P \therefore\ PR^{Y} = 2P \qquad R = 1{\cdot}05 = \downarrow 4879015, = \downarrow r,$$

$$R^{Y} = 2$$

$$\therefore\ \downarrow Yr, = \downarrow 69314718, \text{ and } Yr = 69314718$$

$$Y = \frac{69314718}{r} = \frac{69314718}{4879015} = 14{\cdot}2068404 \text{ years.}$$

When $A = a\downarrow a$, that is, the common number A = the common number represented by the dual log a, multiplied by a. $\downarrow,(A) = \downarrow,a + a$, or the dual log of A = the dual log a plus a,

Thus

$$2688{\cdot}159856 = 2^{4}\ 10^{2}\downarrow 51885180,$$

$$\therefore\ \downarrow,(2688{\cdot}159856) = \downarrow,(2^{4}\ 10^{2}) + 51885180,$$

The use of this is seen in the next example.

Ex. 13. *In how many years will £2221·65592 amount to £5942·81772 at $4\frac{3}{4}$ per cent. per annum, compound interest?*

28. Here R, M, and P are given to find Y.

$$PR^{Y} = M$$

$$R^{Y} = \frac{M}{P} = (m - p)\downarrow(m - p),$$

$$\therefore \quad \downarrow,(R^{Y}) = Yr, = (m - p), + \downarrow, (m - p)$$

$$\therefore \quad Y = \frac{(m - p, + \downarrow(m - p),}{r}$$

$R = 1·0475 = \downarrow 4640641, = \downarrow r,$

$M = 5942·81772 = 2^2\, 10^8 \downarrow 39578905,$
$P = 2221·65592 = 2^1\, 10^8 \downarrow 10510570,$

$$R^{Y} = \frac{M}{P} = \quad 2 \downarrow 29068335, = (m - p)\downarrow(m - p),$$

$$\downarrow(R^{Y}) = \downarrow Yr, = \downarrow 98383053,$$

$$\therefore \quad Y = \frac{98383053}{4640641} = 21·200316 \text{ years.}$$

29. To find the amount when the principal is increased by the interest every year, and another sum at the same time.

If A be the sum added every year, the first A will be at interest Y − 1 years, the second A will be at interest Y − 2 years, and so on;

$\therefore$ the sum of their amount will be

$$AR^{Y-1} + AR^{Y-2} + \dots AR^{Y-Y} = A(R^{Y-1} + R^{Y-2} + \dots 1.)$$

The sum will be $A\frac{R^Y - 1}{R - 1}$, since the terms within the parenthesis form a geometrical progression. But the amount of the principal P in Y years being PR^Y, therefore the whole amount

$$M = PR^Y + A\frac{R^Y - 1}{R - 1}; \quad (1).$$

When $A = P$, then

$$M = P\frac{R^{Y+1} - 1}{R - 1}; \quad (2).$$

When A is not added the last year, then

$$M = PR^Y + AR\frac{R^{Y-1} - 1}{R - 1}; \quad (3).$$

In the last case let $A = P$, then

$$M = PR\frac{R^Y - 1}{R - 1}; \quad (4).$$

If instead of $P = A$, in (1), $P = 0$, then we have the amount of an annuity A, at compound interest, left unpaid for Y, years,

or, $$M = A\frac{R^Y - 1}{R - 1}; \quad (5).$$

If P be the present value of an annuity A for Y years,

then $$PR^Y = A\frac{R^Y - 1}{R - 1}, \text{ or } P = \frac{A}{R - 1}\left\{1 - \frac{1}{R^Y}\right\}; \quad (6).$$

In (6) when Y is infinite, $\frac{1}{R^Y} = 0$, then,

$$P = \frac{A}{R - 1}; \qquad (7).$$

the present value of an annuity to continue for ever, or present value of a *perpetuity* of £A per annum.

Suppose an annuity to be in reversion, that is, not receivable until Z years have elapsed, then the present value for Y + Z years, minus the present value for Z years, gives the present values for Y years after Z years have elapsed, that is

$$P = \frac{A}{R-1}\left\{1 - \frac{1}{R^{Z+Y}}\right\} - \frac{A}{R-1}\left\{1 - \frac{1}{R^Z}\right\} = \frac{1}{R^Z}\,\frac{A}{R-1}\left\{1 - \frac{1}{R^Y}\right\}; \quad (8).$$

Ex. 14. *Suppose £500 put out at compound interest at 4 per cent. per annum, and that £120 is added yearly to the stock; what will be the amount at the end of the 12th year?*

The £120 is not added at the end of the 12th year, as it would not bear interest.

$$\therefore \text{ By (3),} \qquad M = PR^Y + AR\,\frac{R^{Y-1} - 1}{R - 1}$$

$$= PR^Y + A\,\frac{R^Y - R}{R - 1}$$

When P and A are round numbers, it will be found more concise not to take their dual logarithms; in this case the derived form of (3) will be found convenient.

Here

$R = 1{\cdot}04 = \downarrow 3922075, = \downarrow r, \quad P = 500. \quad A = 120. \quad Y = 12.$

$$\therefore\ R^{Y} = \downarrow 12(3922075), = \downarrow Yr, = \downarrow 47064900, = 1{\cdot}60103287$$

$$R^{Y} - R = 1{\cdot}60103287 - 1{\cdot}04 = {\cdot}56103287$$

$$PR^{Y} = 1{\cdot}60103287 \times 500 = 800{\cdot}51639$$

$$M = PR^{Y} + A\,\frac{R^{Y} - R}{R - 1}$$

$$= 800{\cdot}51639 + \frac{120 \times {\cdot}56103287}{1{\cdot}04 - 1}$$

$$= 800{\cdot}51639 + 1683{\cdot}09861$$

$$= \pounds 2483{\cdot}61500$$

Ex. 15. *Find what an annuity of* £50 *will amount to in* 20 *years at* $3\frac{1}{2}$ *per cent. compound interest.*

$$M = \frac{A\,(R^{Y} - 1)}{R - 1}, \text{ from (5) page 42.}$$

$$R = 1{\cdot}035 = \downarrow 0,3,4,5,5,2,4,6, = \overset{8}{\downarrow} 3440145,$$

$$R^{Y} = (1{\cdot}035)^{20} = \overset{8}{\downarrow} 20 \times 3440145, = \overset{8}{\downarrow} 68802900, = \downarrow 7,2,0,9,5,7,0,8,$$

and

$$\downarrow 7,2,0,9,5,7,0,8, = 1{\cdot}98978978;$$

Then

$$\frac{{\cdot}98978978 \times 50}{{\cdot}035} = \pounds 1413{\cdot}9853 = M.$$

These reductions and processes may be more concisely indicated thus, $\downarrow,(R) = \downarrow,(1{\cdot}035) = 3440145,$

$$\downarrow,(R^{Y}) = Y\downarrow,(R) = 20 \times 3440145,$$

$$20 \times 3440145, = 68802900, = \downarrow,(1{\cdot}98978978)$$

Then

$$\frac{{\cdot}98978978 \times 50}{{\cdot}035} = 1413{\cdot}9853$$

30. The work employed to effect the transformations and reductions of the previous Example, without being abridged, may be arranged as follows.

To find a dual number corresponding to 1·035.

$$\downarrow 0,3, = \dot{1}.\dot{0}\dot{3}\dot{0}\dot{3}\overset{\dagger}{0}100\ )\ 1{\cdot}03500000\ (\ 1{\cdot}00456081$$

$$\begin{array}{r} 1{\cdot}03030100 \\ \hline 469900 \\ 412120 \\ \hline 57780 \\ 51515 \\ \hline 6265 \\ 6182 \\ \hline 83 \\ 82 \\ \hline 1 \end{array}$$

To reduce 1·00456081 to a dual number.

$$\begin{array}{lr|r|l}
 & 100 & 4\ 56 & 08\ 1 \\
\overset{8}{\downarrow}4 & - & 4\ 00 & 60\ 0 \\
\hline
 & 100.0\ 55 & 48\ 1 & \\
 & - & 22\ 2 & 4(55{\cdot}4)
\end{array}$$

$$\begin{array}{lr|r|l}
 & 100\ 0 & 55\ 25 & 9 \\
\overset{4}{\downarrow}5 & - & 50\ 01 & 0 \\
\hline
 & 100\ 0 & 05\ 24 & 9 \\
 & & - & 3\quad 5(0{\cdot}5)
\end{array}$$

$$\therefore\quad 1{\cdot}00\,005\,24\,6\,\overset{8}{\downarrow}4,5, = 1{\cdot}00456081$$

For other methods, see "Dual Arithmetic, a New Art," pp. 10 to 28.

It is clear that

$$1{\cdot}00005246 \downarrow 5, = 1{\cdot}00055259$$

and

$$1{\cdot}00055259 \overset{8}{\downarrow} 4, = 1.00456081,$$

$$\therefore \quad 1{\cdot}035 = \downarrow 0,3,4,5,5,2,4,6,$$

To reduce $\downarrow 0,3,4,5,5,2,4,6,$ to a dual logarithm, (12), p. 15.

$$\begin{array}{rr} & \downarrow 0,3,4,5,5,2,4,6, \\ & 99 = 3 \times 33 \\ \hline & 03455345 \\ & 0015200 = 003040 \times 5 \\ \hline \downarrow, 0,3,4,5,5,2,4,6, = & 3440145, \end{array}$$

To find the natural number answering to

$$3440145, \times 20 = 68802900,$$

see Rule, (15), p. 21.

$$\begin{array}{rl} & 68802900, \\ \downarrow, (\tfrac{1}{2}) = & '69314718 \\ \hline & '511818 \\ & 250 = u_8(50,) \\ \hline & 511568 = '0'0'5'1'1'5'6'8 \uparrow \end{array}$$

$$\therefore \quad '0'0'5'1'1'5'6'8' \uparrow 2 = 68802900, = \downarrow, (1{\cdot}98978978)$$

It may also be shown (13), p. 17, that

$$68802900, = \downarrow, 7,2,0,9,5,7,0,8,$$

$$\text{and} \quad \downarrow 7,2,0,9,5,7,0,8 = 1{\cdot}98978978$$

31. The next examples, numbered I. II. III. &c. may be considered an amplification, and are selected to illustrate other simple but important dual reductions which have often to be made.

PRELIMINARY REDUCTIONS OFTEN REQUIRED IN CALCULATING THE ROOTS OF EQUATIONS.

I. If $\downarrow 7,0,0,0,0,0,0,0,0, = R^{20}$; then $R = \downarrow 0,3,3,5,0,9,0,7,$

Reduction, (12).

$$
\begin{array}{rll}
 & 2\,1\,7\,1\,2\,6 & = 7 \times 31018 \\
 & \downarrow 7,0,0,0,0,0,0,0,0, & \\
\text{Subtract} & 3\,5 & = 7 \times 5 \ldots\ldots \\
\hline
20\,) & 6\,6\,7\,1\,7\,1\,2\,6, & \\
\hline
 & 0\,3\,3\,3\,5\,8\,5\,6, & \\
 & 0\,0\,1\,5\,1\,5\,0 & = 003030 \times 5 \\
\hline
 & 0\,3\,3\,5\,1\,0\,0\,6 & \\
 & 9\,9 & = 3 \times 33 \\
\hline
\end{array}
$$

$\therefore$ (13) $R = \downarrow 0,3,3,5,0,9,0,7,$

It is scarcely necessary to add that,

Reduction, (12)

signifies that the succeeding reduction is made according to the principles explained in article (12); and that $\therefore$ (13), denotes therefore by article (13).

II. If $R^{20} = \downarrow 7,0,0,0,$ then $R = \downarrow 0,3,5,$ nearly; this result may be found by mere inspection, for taking the dual as a common number, we have

$$\frac{7}{20} = \cdot 035 \text{ which is a little over } \cdot 0335$$

$$\therefore \quad \downarrow 0,3,5, = \downarrow 0,3,3,5, \ldots\ldots \text{ nearly.}$$

The succeeding remarks should be particularly observed.

32. ☞ Dual developments never result in approximate values; however unlike equal dual forms may appear, every form

of the required value is true to the designed degree of accuracy, which may be as great as we please. From the flexibility of dual numbers we are not obliged to resort to trial and error, nor are we confined to one set of numbers and particular narrow intervals to exhibit required results. ☞ 28.

III. In the above example

$$R = \downarrow 0,3,3,5,0,9,0,7, = 1\cdot03392116$$

and

$$R = '0'0'1'4'8'9'4'3 \downarrow 0,3,5, = 1\cdot03392116$$

also

In finding $\downarrow 0,3,3,5,0,9,0,7, = 1\cdot03392116$ in a direct manner without being obliged to retrace our steps, $1\cdot03392116$ is not found by approximation because $'0'0'1'4'8'9'4'3 \downarrow 0,3,5,$ is also found equal to $1\cdot03392116$ by a like direct procedure.

IV. If $\downarrow 0,0,9,0,0,0,0,0 = R^{20}$, then $R = \downarrow 0,0,0,4,4,9,7,8,$

Reduction. (12).

$$\begin{array}{rl} & \downarrow 0,0,9,0,0,0,0,0, \\ \text{Sub.} & \underline{\;000045\;} \\ 20\,) & \underline{008995\,50} \\ & \downarrow 0,0,0,4,4,9,7,8 \end{array}$$

$\frac{009}{20} = 00045,$ which is not much greater than 0044978.

33. When the first three dual digits are zeros, or $\downarrow 0,0,0\ldots\ldots$ the remaining five dual digits may be treated, and reduced as if they were common numbers; for example, if $\downarrow 0,0,0,9,8,0,2,8, = R^3$, then $R = \downarrow 0,0,0,3,2,6,7,6,$ See Article (9).

Reduction.

$$\begin{array}{rl} 3\,) & \underline{\downarrow 0,0,0,9,8,0,2,8,} \\ & \downarrow 0,0,0,3,2,6,7,6, \end{array}$$

This property may be generally established thus, the dual logarithm of $\downarrow 0,0,0,u_4,u_5,u_6,u_7,u_8,$

written $\downarrow, 0,0,0,u_4,u_5,u_6,u_7,u_8,$ (See page 9).

is equal $10000u_4 + 1000u_5 + 100u_6 + 10u_7 + u_8$

and is the natural number expressed by the dual digits taken as common digits.

V. If $R^{\frac{2}{3}} = \downarrow 2,3,4,5,6,7,8,9,$ then $R^{\frac{5}{7}} = \downarrow 2,5,0,7,3,9,3,2,$

Reduction. See Articles (12) (13).

	62036	+ twice 31018
	$\downarrow$ 2,3,4,5,6,7,8,9,	
Subtract	101520	– five times 2,03,04,0
	99	+ three times 33
	22503724	
	3	
2)	67511172	
	33755586	
	5	
7)	168777930	
	24111133	
+	10	+ twice 5 . . . . .
	25111133	
	62036	– twice 31018
	25049097	
+	2500	+ five times 5 . . .
	25074097	
	165	– five times 33.
	$\downarrow$ 2,5,0,7,3,9,3,2,	

34. These reductions are so readily effected, and so fully illustrated here and elsewhere, that, in future, such trifling calculations, in most instances, will be omitted, and in such cases, for example, we shall say,

if $R^{20} = \downarrow 7,$ $\quad\therefore\quad R = \downarrow 0,3,3,5,0,9,0,7,$;

if $R^{\frac{3}{4}} = \downarrow 2,3,4,5,6,7,8,9,$ $\quad\therefore\quad R^{\frac{4}{5}} = \downarrow 2,5,0,7,3,9,3,2,$; &c.

It is a very easy dual operation to find $R^{\frac{m}{n}}$ when $R^{\frac{p}{q}}$ is given, yet a practical solution of this simple problem, without the use of tables, defied the combined skill of mathematicians before the Author, Oliver Byrne, discovered and developed dual arithmetic.

QUESTIONS RELATING TO INTEREST AND ANNUITIES CONTINUED.

Ex. 16. *Suppose an annuity of £50 to amount to £1413·98528 in 20 years, what is the rate per cent. compound interest?*

From (5), page 42,

$$M = A\,\frac{R^{Y} - 1}{R - 1};$$

or

$$MR - AR^{Y} = M - A$$

$$\therefore\quad 1413{\cdot}98528R - 50R^{20} = 1363{\cdot}98528; \quad (K).$$

$R = 1$, will satisfy equation (K), but this value of R is not admissible, since R is always greater than 1. Hence another value of R must be found, such that 50 times R in the 20th power taken from 1413·98528 will leave a remainder = 1363·98528, (K). Many dual numbers may be found, and each reducible to $R = 1{\cdot}035$, which will also satisfy equation (K); one set of these dual numbers will be presently found, showing that when

$$R^{20} = \downarrow 7,2,0,9,5,5,6,3, \text{ then } R = \downarrow 0,3,4,5,5,2,3,6, = 1{\cdot}035$$

35. It has been proved in particular cases (27) (I.) (II.) &c. and it will be generally established hereafter, that if $R^{n} = \downarrow u \ldots$ and $R^{m} = \downarrow w \ldots$ then the ratio of $\downarrow u \ldots$ to $\downarrow w \ldots$ taken as natural numbers approaches the ratio of n to m. This knowledge would be of little value unless we also had the power, when R^{n} is given or assumed, to determine R^{m} true to the limit of the designed degree of accuracy, and besides every path pointed out by the ratio of n to m should lead in a direct way, without deviation, to the exact value sought. These demands are fully supplied by dual arithmetic. Nor is this all, for we may take dual digits much greater or much less than any particular digits pointed out by the ratio of n to m, and yet obtain, without guessive artifices, a result as near the truth as we please. Since this method gives the same result, as near the truth as we please, by several direct processes, it presents a series of direct operations, and not a succession of approximate trials. A method may be direct, and yet give results that continually approximate to correct results.

Returning to equation (K) which may be put under the form

$$ax - bx^{20} = c$$

In which

$$a - b = c$$

The fraction that $\frac{a}{20}$ is of b will point out the first dual digit.

$$\frac{a}{20} = \frac{1413 \ldots\ldots}{20} = 70.$$

$$b \div \frac{a}{20} = \frac{50 \ldots}{70 \ldots} = \cdot 7 \ldots$$

If $\quad R^{20} = \downarrow 7, \quad$ then $\quad R = \downarrow 0{,}3{,}3{,}5{,}0{,}9{,}0{,}7, \quad$ (34).

$\downarrow 0{,}3{,}3{,}5{,}0{,}9{,}0{,}7,$	(1)	$\downarrow 7{,}0{,}0{,}0{,}0{,}0{,}0{,}0,$	(1)
$+\ 1413{\cdot}98528$	(2)	$-\ 50{\cdot}0000000$	(2)
$+\ 1461{\cdot}94934$	(3)	$-\ 97{\cdot}4358550$	(3)

(2) multiplied by (1) produces (3). This arrangement is maintained throughout. See "Dual Arithmetic, a New Art," pp. xxvi. and 173.

In finding a second convenient dual digit for a dual value of R^{20}, many of the succeeding figures might have been omitted.

+ 73·	$\frac{1}{20}$ of		+ 1461·94934	
− 97·	once		− 97·435855	
− 24			1364·513485	take
			1363·985280	from (K).
		− 24)	− ·528	(+ $\downarrow$0,2,
			− 48·	

36. It may be observed that the method above instituted resembles common division, but the quotient figure is a dual digit, and may be in excess or defect, greater or less than 9, without involving error; the case is otherwise with common division.

If $r^{20} = \downarrow 0,2,$ then $r = \downarrow 0,0,0,9,9,5,0,3,$ (34).

The remainder of the process employed to find R^{20} and then (34) R may be arranged in the succeeding order.

Mult. by	$\downarrow$0,0,0,9,9,5,0,3,	$\downarrow$0,2,
	+ 1461·94934	− 97·4358550
gives	1463·40468	− 99·3943157

+ 73·17	$\frac{1}{20}$ of	+ 1463·40	468	
− 99·39	once	− 99·39	43157	
− 26·22)		1364·01	03643	take
		1363·98	530	from (K)
		− 2	506	(+ $\downarrow$0,0,0,9,
		− 2	358	

If $r^{20}, = \downarrow 0,0,0,9,$ then $r, = \downarrow 0,0,0,0,4,5,0,0,$ (34).

Mult. by	$\downarrow$0,0,0,0,4,5,0,0,	$\downarrow$0,0,0,9,
	+ 1463·40468	− 99·3943157
gives	+ 1463·47055	− 99·4838064

$$
\begin{array}{lrr|ll}
+73\cdot17 & & +1463\cdot470 & 55 & \\
-99\cdot48 & & -\ \ 99\cdot483 & 8064 & \\
\hline
-26\cdot31 & & +1363\cdot986 & 7436 & \text{take} \\
 & & +1363\cdot985 & 28 & \text{from (K)} \\
\hline
 & -2\dot{6}31) & -1 & 4636(\downarrow 0,0,0,0,5,5,6,3, & \\
 & & 1 & 3155 & \\
\cline{4-4}
 & & & 1481 & \\
 & & & 1316 & \\
\cline{4-4}
 & & & \ \ 165 & \\
 & & & \ \ 158 & \\
\cline{4-4}
 & & & \ \ \ \ 7 & \\
\end{array}
$$

$$\therefore\ R^{20} = \downarrow 7,2,0,9,5,5,6,3,$$

and $\therefore\ R = \downarrow 0,3,4,5,5,2,3,6, = 1\cdot03499987 = 1\cdot035$ nearly.

The rate per cent. may be said to be $3\frac{1}{2}$.

If our decimals had been carried out sufficiently far, we should have obtained 1·035 instead of 1·03499987. This example is the reverse of example 15, page 44.

Ex. 17. *An annuity of* £140 *left unpaid for* 33 *years amounted to* £16650·71552, *compound interest; what was the rate per cent.?*

According to the formula employed in the last problem

$$16650\cdot71552\,R - 140\,R^{33} = 16510\cdot71552; \text{ (K)}$$

$R = 1$ satisfies (K), but this value of R is not admissible, for R is always greater than 1. If the operation of extracting the next root to 1, be commenced with 6, 7, 8, or 9, respectively,

then it will be found that

$$\left.\begin{aligned} R^{33} &= 6\downarrow 4,6,0,0,3,5,2,5,\\ &= 7\downarrow 2,9,6,6,5,6,9,4,\\ &= 8\downarrow 1,5,8,2,3,8,5,2,\\ &= 9\downarrow 0,3,5,6,6,4,3,5,\end{aligned}\right\} = 9{\cdot}325.$$

It is evident that 8 may be substituted for R^{33}, since 8 times $140 = 1120$, which approaches ·07 times 16650. How to find such limits of the values of unknown quantities will be discussed presently. However, in this equation (K),

the 33rd root of $6 = \downarrow 0,5,\ldots$ the 33rd root of $7 = \downarrow 0,6,\ldots$
the 33rd root of $8 = \downarrow 0,6,\ldots$ & the 33rd root of $9 = \downarrow 0,7,\ldots$

Hence, a mistake can scarcely be made, even by those ignorant of the theory of equations, so great is the range of convenient dual forms which the required value may be made to assume.

$$\begin{array}{l} 33\,)\,\underline{\downarrow 2\,0\,7\,9\,4\,4\,1\,5\,4,} = \text{dual log. of } 8\\ \quad\;\; 0\,6\,3\,0\,1\,3\,3\,8\\ \downarrow\; 0,6,3,3,1,2,9,0 = 33\text{rd root of } 8.\end{array}$$

$$\begin{array}{lll} \text{Multiply by} & \downarrow 0,6,3,3,1,2,9,0 & 8\downarrow\\ & +\ 16650\ldots\ldots & -\ 140\ldots\ldots\\ \text{gives} & +\ 17733\ldots\ldots & -\ 1120\ldots\ldots \end{array}$$

$$\begin{array}{lcr|ll} +\ \ 537 & \tfrac{1}{33}\text{ of} & +\ 177 & 33\ldots\ldots & \\ -\ 1120 & \text{once} & -\ \ \ 11 & 20\ldots\ldots & \\ \hline -\ \ 583\ ) & & 166 & 13\ldots\ldots & \text{take}\\ & & 165 & 10\ldots\ldots & \text{from (K)}\\ \hline & & -\ \ \ 1 & 03 & (+\downarrow 1,\\ & & - & 58 & \end{array}$$

If $\quad r^{33} = \downarrow 1,$ then $r = \downarrow 0,0,2,8,8,9,1,9,$ (34).

Mult. by	$\downarrow 0,0,2,8,8,9,1,9,$	$\downarrow 1,$
	$17733 \ldots\ldots$	$-\ 1120 \ldots\ldots$
Result	$17784 \ldots\ldots$	$-\ 1232 \ldots\ldots$

$$\begin{array}{rl|l} +\ 538\ldots & \tfrac{1}{33} \text{ of} & +\ 1778|4\ldots\ldots \\ -\ 1232\ldots & \text{once} & -\ \ \ 123|2\ldots\ldots \\ \hline -\ \ 694\,) & & \ \ 1655|2\ldots\ldots \text{ take} \\ & & \ \ 1651|0\ldots\ldots \text{ from (K)} \\ \hline & & \ \ -4|2\ldots\ldots \quad (\downarrow 0,5, \end{array}$$

If $\quad r_{,}^{33} = \downarrow 0,5,$ then $r_{,} = \downarrow 0,0,1,5,0,8,1,3,$ (34).

Mult. by	$\downarrow 0,0,1,5,0,8,1,3,$	$\downarrow 0,5,$
	$+\ 1778498\ldots$	$-\ 123200$
Result	$+\ 1781182\ldots$	$-\ 129484$

$$\begin{array}{rl|l} +\ 539{\cdot}7 & \tfrac{1}{33} \text{ of} & +\ 17811|82\ldots \\ -\ 1294{\cdot}8 & \text{once} & -\ \ \ 1294|84\ldots \\ \hline -\ \ 755\ 1) & & \ \ 16516|98\ldots \text{ take} \\ & & \ \ 16510|71\ldots \text{ from (K)} \\ \hline & & \ \ -\ 6|27\ldots \\ & & \ \ -\ 6|04\ldots \quad (\downarrow 0,0,8,2, \\ \hline & & \ \ \ \ \ 23 \end{array}$$

The root being so far determined by contracted operations, let

$$R^{33} = 8 \downarrow 1,5,8,2, \text{ then } R = \downarrow 0,6,7,9,5,9,0,8, \qquad (34).$$

The succeeding operation by using the coefficients of R and R^{33} in the original equation (p. 53) is independent of those above employed to show that $8 \downarrow 1,5,8,2, \ldots .$ is a convenient dual form of R^{33}. What follows, not only determines R to the designed degree of accuracy, but also proves the preliminary calculations.

Mult. by	$\downarrow 0,6,7,9,5,9,0,8,$	$8 \downarrow 1,5,8,2,$
	$+\ 16650{\cdot}71552$	$-\ \ 140{\cdot}000000$
Result	$+\ 17816{\cdot}24553$	$-\ 1305{\cdot}500527$

$$
\begin{array}{rl}
+ & 539{\cdot}886 \\
- & 1305{\cdot}500 \\
\hline
- & 765{\cdot}614
\end{array}
\quad \tfrac{1}{33} \text{ of } \text{once} \quad
\begin{array}{rr|l}
+ & 17816{\cdot}24 & 553 \\
- & 1305{\cdot}50 & 053 \\
\hline
 & 16510{\cdot}74 & 500 \text{ take} \\
 & 16510{\cdot}71 & 552 \text{ from (K)} \\
\hline
7656\,) & -\;2 & 948 \qquad (\downarrow 0,0,0,0,3,8,5,2, \\
 & 2 & 296 \\
\cline{3-3}
 & & 652 \\
 & & 612 \\
\cline{3-3}
 & & \;\;40 \\
 & & \;\;38 \\
\cline{3-3}
 & & \;\;\;\;2
\end{array}
$$

$$\therefore \quad R^{33} = 8 \downarrow 1,5,8,2,3,8,5,2, \qquad \therefore \quad R = \downarrow 0,6,7,9,6,0,2,5, = 1.07;$$

and $\therefore$ 7 is the rate per cent.

In Examples 16 and 17, dual methods of calculating unknown but well-defined magnitudes are applied without referring to the general dual system of solving equations hereafter discussed.

Ex. 18. *If the yearly rent of a freehold be £200, what is its present value* at $5\frac{1}{2}$ *per cent. compound interest?*

From (7), page 43,

$$P = \frac{A}{R - 1};$$

$$\therefore \quad \frac{200}{\cdot 055} = 3636\tfrac{4}{11}, \text{ the present value.}$$

This example is introduced for the sake of uniformity, the required result is found by common division. The dual method is chiefly applied where logarithms have been found peculiarly serviceable, and in cases when neither logarithmic nor common arithmetical operations will apply, as in Examples 16 and 17.

Ex. 19. *Required the present value of an annuity of* £140, *which is to continue* 33 *years at* 7 *per cent. compound interest.*

The amount is £16650·71552. See Example 17.

$$\therefore \quad (33) \quad PR^{Y} = M = 16650{\cdot}71552$$

$$R = 1{\cdot}07 = \downarrow 6765871,$$

$$\therefore \quad R^{Y} = \downarrow 33 \times 6765871, = \downarrow 223273743,$$

$$M = 10^4 \downarrow 50986811, = 10^3 \downarrow 281245320,$$

$$P = \frac{M}{R^{Y}} = \frac{\mathrm{m} \downarrow m,}{\downarrow Yr,} = \frac{10^3 \downarrow 281245320,}{\downarrow 223273743,} = 10^3 \downarrow 57971577,$$

$$\downarrow 57971577, = 1{\cdot}78553179$$

$$\therefore \quad P = 1785{\cdot}53179$$

Ex. 20. *In how many years will an annuity of* £50 *amount to* £2000, *at* $4\frac{3}{8}$ *per cent. per annum, compound interest?*

From (5), page 42,

$$M = A \frac{R^{Y} - 1}{R - 1},$$

$$\therefore \quad \downarrow Yr, = R^{Y} = \left(1 + \frac{(R-1)M}{A}\right), \text{ which put } = \downarrow s,$$

$$\downarrow, (R) = r, \qquad \downarrow, (R^{Y}) = Yr,$$

$$\therefore \quad Yr, = s, \quad \text{or} \quad Y = \frac{s}{r}$$

$$R = 1{\cdot}04375,$$

$$1 + \frac{M(R-1)}{A} = 1 + \frac{2000(\cdot 04375)}{50} = 2{\cdot}75$$

$$2{\cdot}75 = 2 \downarrow 31845377, = \downarrow 101160095, = \downarrow s,$$

and $$1{\cdot}04375 = \downarrow 4282001, = \downarrow r,$$

$$Y = \frac{s}{r} = \frac{101160095}{4282001} = 23{\cdot}6245 \text{ years.}$$

$$\text{or } Y = 23\tfrac{5}{8} \text{ years nearly.}$$

OPERATIONS INDICATED BY THE SIGNS $\downarrow$ AND $\rightarrow$.

37. Before proceeding further, it is necessary to explain how the dual sign of addition ($\downarrow$), and the dual sign of subtraction ($\rightarrow$), of the ascending branch, differ from the ordinary signs of addition and subtraction, and how these new signs are operated with.

Let $32{\cdot}576 \downarrow 2,5,1,3,0,0,0,0,$ be respectively represented by $A \downarrow u_1, u_2, u_3, \ldots\ldots$

Then

$$\begin{array}{rccccccccccl}
A = & 3 & 2 & 5 & 7 & 6 & 0 & 0 & 0 & 0 & & \\
 & & 6 & 5 & 1 & 5 & 2 & 0 & 0 & 0 & & \\
 & & & 3 & 2 & 5 & 7 & 6 & 0 & 0 & & \\
\hline
A \downarrow u_1, = & 3 & 9 & 4 & 1 & 6 & 9 & 6 & 0 & 0 & . & \text{which put} = A_1 \\
 & & 1 & 9 & 7 & 0 & 8 & 4 & 8 & 0 & . & \\
 & & & & 3 & 9 & 4 & 1 & 7 & 0 & . & \\
 & & & & & & 3 & 9 & 4 & 2 & . & \\
 & & & & & & & & 2 & 0 & . & \\
\hline
A \downarrow u_1, u_2, = & 4 & 1 & 4 & 2 & 7 & 6 & 2 & 1 & 2 & & \text{which put} = A_2 \\
 & & & & 4 & 1 & 4 & 2 & 7 & 6 & & \\
\hline
A \downarrow u_1, u_2, u_3, = & 4 & 1 & 4 & 6 & 9 & 0 & 4 & 8 & 8 & & \text{which put} = A_3 \\
 & & & & 1 & 2 & 4 & 4 & 0 & 7 & & \\
 & & & & & & & & 1 & 2 & & \\
\hline
A \downarrow u_1, u_2, u_3, u_4, = & 4 & 1{\cdot}4 & 8 & 1 & 4 & 9 & 0 & 7 & & & \text{which put} = A_4
\end{array}$$

Then, by employing the dual sign of addition ($\dotplus$), this continuous and well-known process may be indicated as follows:—

$$
\begin{array}{llll}
A \downarrow u_1 & = A \dotplus u_1 A & = A\,(1 \dotplus u_1) = A_1\,; \\
A \downarrow u_1, u_2, & = A_1 \downarrow 0, u_2, & = A_1 \dotplus u_2 A_1 = A_2\,; \\
A \downarrow u_1, u_2, u_3, & = A_2 \downarrow 0, 0, u_3, & = A_2 \dotplus u_3 A_2 = A_3\,; \\
A \downarrow u_1, u_2, u_3, u_4, & = A_3 \downarrow 0,0,0,0,u_4, & = A_3 \dotplus u_4 A_3 = A_4\,; \\
\quad \&c. & \quad \&c. & \quad \&c.
\end{array}
$$

$$
\begin{array}{lll}
\downarrow 0, u_2, & \text{is written } \downarrow u_2, & \text{or } \downarrow^2 u, \\
\downarrow 0, 0, u_3, & \text{is written } \downarrow u_3, & \text{or } \downarrow^3 u, \\
\downarrow 0, 0, 0, u_4, & \text{is written } \downarrow u_4, & \text{or } \downarrow^4 u, \\
\quad \&c. & \quad \&c. &
\end{array}
$$

38. The units $u_1, u_2, u_3, \ldots.$ in conjunction with $\dotplus$, may be operated with in any order whatever, provided that all the units are incorporated.

$$
\begin{array}{llll}
A \downarrow 0,0,u_3 & = A \dotplus u_3 A & = A\,(1 \dotplus u_3) & \text{which put} = B_1\,; \\
A \downarrow u_1, 0, u_3, & = B_1 \downarrow u_1, & = B_1 \dotplus u_1 B_1 & \text{which put} = B_2\,; \\
A \downarrow u_1, 0, u_3, u_4, & = B_2 \downarrow^4 u_4, & = B_2 \dotplus B_2 u_4 & \text{which put} = B_3\,; \\
A \downarrow u_1, u_2, u_3, u_4, & = B_3 \downarrow^2 u_2 & = B_3 \dotplus u_2 B_3 & \text{which put} = B_4\,; \\
\quad \&c. & \quad \&c. & \quad \&c. &
\end{array}
$$

Then $$A_4 = B_4.$$

Again,

$$
\begin{array}{ll}
A \downarrow u_1, & = A \dotplus u_1 A = A\,(1 \dotplus u_1) = A_1\,; \\
A \downarrow u_1, u_2, & = A_1 \dotplus u_2 A_1 = A_1\,(1 \dotplus u_2) = A_2\,; \\
& = (A \dotplus u_1 A) \dotplus u_2\,(A \dotplus u_1 A) \\
& = A\,(1 \dotplus u_1) \dotplus u_2 A\,(1 \dotplus u_1) \\
& = A\,[(1 \dotplus u_1) \dotplus u_2\,(1 \dotplus u_1)]
\end{array}
$$

$$
\begin{array}{ll}
A \downarrow u_1, u_2, u_3, & = A_2 \downarrow^3 u_3, = A_2 \dotplus u_3 A_2 = A_3\,; \\
& = \{A\,[(1 \dotplus u_1) \dotplus u_2\,(1 \dotplus u_1)]\} \dotplus u_3\,\{A\,[(1 \dotplus u_1) \dotplus u_2\,(1 \dotplus u_1)]\} \\
& = A\{[(1 \dotplus u_1) \dotplus u_2\,(1 \dotplus u_1)] \dotplus u_3\,[(1 \dotplus u_1) \dotplus u_2\,(1 \dotplus u_1)]\}\,; \\
& \qquad \&c. \qquad \&c.
\end{array}
$$

39. The sign of dual subtraction ($\rightarrow$), ascending branch.

$$\frac{A}{\downarrow u_1,} = A\downarrow\bar{u}_1, = A \rightarrow u_1 A = A(1 \rightarrow u_1) \text{ which put } = D_1;$$

$$\frac{A}{\downarrow u_1,u_2,} = A\downarrow\bar{u}_1,\bar{u}_2, = D_1\downarrow^2\bar{u}_2, = D_1 \rightarrow u_2 D_1 = D_1(\rightarrow u_2) \text{ which put} = D_2;$$

$$\frac{A}{\downarrow u_1,u_2,u_3,} = A\downarrow\bar{u}_1,\bar{u}_2,\bar{u}_3, = D_2\downarrow^3\bar{u}_3, = D_2 \rightarrow u_3 D_2 \text{ which put } = D_3;$$

&c. &c.

Develop $A\downarrow u_1,u_2,u_3,\bar{u}_4, \ldots$ in a form involving both $\mathbin{\downarrow\!\!\!+}$ and $\rightarrow$.

$$A_1\downarrow u_1, = A_1 \mathbin{\downarrow\!\!\!+} u_1 A_1 = A_2;$$

$$A_1\downarrow u_1,\bar{u}_2, = A_2\downarrow^2 u_2, = A_2 \rightarrow u_2 A_2 = A_3;$$

$$A_1\downarrow u_1,\bar{u}_2,u_3, = A_3\downarrow^3 u_3, = A_3 \mathbin{\downarrow\!\!\!+} u_3 A_3 = A_4;$$

$$A_1\downarrow u_1,\bar{u}_2,u_3,\bar{u}_4, = A^4\downarrow\bar{u}_4, = A_4 \rightarrow u_4 A_4 = A_5;$$

&c. &c.

40. The dual sign of plus or minus for the ascending branch is written $\mathbin{\downarrow\!\!\!+}$. When the dual digits $u_1, u_2, u_3, \ldots$ are either plus or minus, one of the many developments of $A\downarrow u_1, u_2, u_3, \ldots$ will be

$$\{[(A \mathbin{\downarrow\!\!\!+} u_1 A) \mathbin{\downarrow\!\!\!+} u_2(A \mathbin{\downarrow\!\!\!+} u_1 A)] \mathbin{\downarrow\!\!\!+} u_3[(A \mathbin{\downarrow\!\!\!+} u_1 A) \mathbin{\downarrow\!\!\!+} u_2(A \mathbin{\downarrow\!\!\!+} u_1 A)]\} \mathbin{\downarrow\!\!\!+}$$
$$u_4\{[(A \mathbin{\downarrow\!\!\!+} u_1 A) \mathbin{\downarrow\!\!\!+} u_2(A \mathbin{\downarrow\!\!\!+} u_1 A)] \mathbin{\downarrow\!\!\!+} u_3[(A \mathbin{\downarrow\!\!\!+} u_1 A) \mathbin{\downarrow\!\!\!+} u_2(A \mathbin{\downarrow\!\!\!+} u_1 A)]\} \mathbin{\downarrow\!\!\!+} \&c.$$

which is more concisely expressed under the form

$$A\{[(1 \mathbin{\downarrow\!\!\!+} u_1) \mathbin{\downarrow\!\!\!+} u_2] \mathbin{\downarrow\!\!\!+} u_3\} \mathbin{\downarrow\!\!\!+} u_4 \ldots\ldots\ldots\} \qquad (Z).$$

41. The development (Z) may be given under as many different forms as there are variations in the permutation of all the dual digits taken together. When eight consecutive ascending positive dual digits are combined, (Z) may be given under 40320 forms, each expressing the same result, for $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$. In the latter case, when $A = 1$, (Z) may be given under the form

$$1 \mathbin{\downarrow\!\!\!+} [u_8 \mathbin{\downarrow\!\!\!+} [u_7 \mathbin{\downarrow\!\!\!+} [u_6 \mathbin{\downarrow\!\!\!+} [u_5 \mathbin{\downarrow\!\!\!+} [u_4 \mathbin{\downarrow\!\!\!+} [u_3 \mathbin{\downarrow\!\!\!+} [u_2 \mathbin{\downarrow\!\!\!+} [u_1 \qquad (Y).$$

Suppose each of the dual digits to be less than 10, and (Y) limited to eight places of decimals, then (Y) becomes

$$1 + [u_8 + [u_7 + [u_6 + [u_5 \underset{\downarrow}{+} [u_4 \underset{\downarrow}{+} [u_3 \underset{\downarrow}{+} [u_2 \underset{\downarrow}{+} [u_1 =$$
$$(10000u_5u_6u_7u_8) \underset{\downarrow}{+} [u_1 \underset{\downarrow}{+} [u_2 \underset{\downarrow}{+} [u_3 \underset{\downarrow}{+} [u_4$$

In future each dual digit is supposed to be less than 10 if the contrary be not specified.

42. Coincidence of the corresponding values of the 5th, 6th, 7th, and 8th dual digits in a tabulated form; $u_1=0$ $u_2=0$ $u_3=0$ $u_4=0$.

	Natural Numbers.	Dual Numbers.	Dual Logarithms.
Particular case	1·00004763	$\downarrow$ 0,0,0,0,4,7,6,3,	4763,
General form	$1\cdot0000u_5u_6u_7u_8$	$\downarrow$ $0,0,0,0,u_5,u_6,u_7,u_8,$	$u_5u_6u_7u_8,$

$(1000)u_5 + (100)u_6 + (\cdot10)u_7 + u_8$ is represented by $u_5u_6u_7u_8,$

$100000000 + (1000)u_5 + (100)u_6 + (10)u_7 + u_8$ is represented by

$$10000u_5u_6u_7u_8$$

No error can be involved through considering $10000u_5u_6u_7u_8$ a whole number while being operated upon.

43. To find the natural number corresponding to a dual number of the form $\downarrow$ $0,0,0,u_4,u_5,u_6,u_7,u_8,$

Let the operative numbers or binomial coefficients

for the dual digit u_4 be represented by 1 u_4 u_4' u_4'' . . .

for the dual digit u_3 be represented by 1 u_3 u_3' u_3'' . . .

&c. &c.

Corresponding values of the 4th, 5th, 6th, 7th and 8th dual digits in a comparative tabulated form, when $u_1 = 0$ $u_2 = 0$ $u_3 = 0$.

	Natural Numbers.	Dual Numbers.	Dual Logarithms.
Particular case	100054763 $10 = u_4'$ $2 = 5(\cdot47)$ ——— 100054775	$\downarrow$ 0,0,0,5,4,7,6,3,	54763,
General form	$1000u_4u_5u_6u_7 \mid u_8,$ $+ \mid u_4'$ $+ \mid u_4\ (\cdot u_5u_6)$ ——— $1000u_4u_5u_6\,p\,q$	$\downarrow$ $0,0,0,u_4,u_5,u_6,u_7,u_8,$	$u_4u_5u_6u_7u_8,$

In this case (Y) becomes $(10000u_5u_6u_7u_8) \downarrow [u_4 =$

$10000u_5u_6u_7u_8 + u_4(10000) + u_4(\cdot u_5u_6) + u_4' =$

$10000u_4u_5u_6u_7u_8 + u_4' + u_4'(\cdot u_5u_6)$

Hence the natural number is equal to the dual number, taken as a natural number, + the third operative number for u_4 + the nearest whole number to u_4 multiplied by the decimal u_5u^6. This rule is readily reversed. Hence

44. To find the dual number and the dual logarithm of a natural number of the form $1 \cdot 000u_4u_5u_6p\,q$.

RULE.

Subtract the third operative number for the digit u_4 and the nearest whole number to $u_4 \times$ decimal $(\cdot u_5u_6)$, the remainder is the required dual number when $\downarrow$ is put for 1. The corresponding dual logarithm is expressed by the five dual digits thus obtained.

Examples.

Ex. 1. *Find the dual number and dual logarithm corresponding to the natural number* 1·00054775.

The third coefficient for 5 is 10; the coefficients or operative numbers for 5 being 1 5 10 10 5 1.

$$5(\cdot 47) = 2\cdot \text{ nearly.}$$

$$\begin{array}{r} 100054775 \\ 12 \\ \hline \end{array} \quad = 10 + 2$$

Dual number = $\downarrow$ 0,0,0,5,4,7,6,3,

$\therefore$ (16) (page 23)$\downarrow$, (1·00054775) = 54763, = $\downarrow$, 0,0,0,5,4,7,6,3,

Ex. 2. *Find the dual number and dual logarithm of the natural number* 1·00078987.

The third coefficient for 7 is 21, the operative numbers being

$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1;$$

$$7(\cdot 89) = 6\cdot \text{ nearly.}$$

$$\begin{array}{l} 100078987 \\ \qquad\qquad 27 = 21 + 6 \\ \hline \downarrow 0,0,0,7,8,9,6,0, \text{ dual number.} \\ \qquad 78960, \text{ dual logarithm.} \end{array}$$

Natural number.	Dual number.	Dual logarithm.
1·00078987	↓ 0,0,0,7,8,9,6,0,	78960,

45. Let each of the dual digits be less than 10, (Y) limited to nine places of figures, and $u_1 = 0$, $u_2 = 0$; then (Y) becomes

$$(10000u_5u_6u_7u_8) \downarrow [u_4 \downarrow [u_3 \qquad (X).$$

In accordance with (41), (X) may be put under the form

$$(1000u_4u_5u_6pq) \downarrow [u_3 =$$
$$(1000u_4u_5u_6pq) + u_3(1000u_4u_5\cdot u_6) + u_3'\,100\cdot 0u_4) + u_3''(\cdot 1) =$$
$$(100u_3u_4u_5u_6pq) + u_3'(100) + u_3''(\cdot 1) + u_3(u_4u_5\cdot u_6) + u_3'(\cdot 0u_4)$$

$u_3'(\cdot 0u_4)$ seldom amounts to a unit, and in most cases may be neglected. In reversing the process $u_3(u_4u_5\cdot u_6)$ must be put under the form

$$u_3(u_4 0\cdot) + u_3(u_5\cdot u_6).$$

46. To find the dual number and dual logarithm answering to a natural number of the form $1\cdot 00u_3u_4rspq$.

RULE.

From the given number subtract the natural number corresponding to $\downarrow 0,0,u_3$; from the remainder take $u_3(u_4 0\cdot)$ and $u_3(u_5$ decimal $u_6)$, and also $u_3'(\cdot 0u_4)$ when it amounts to a whole

number; the last remainder is of the form $u_4 u_5 u_6 pq$, which may be reduced to a dual number by the Rule (44). $\downarrow$ being put for 1.

Examples.

Ex. 1. *Find the dual number and dual logarithm answering to* 1·00789179.

Given number	100|789|179	
$\downarrow$0,0,7,	100|702|104	$\downarrow 0,0,u_3,$
	$\downarrow$00|087|075	
7 (80·)	· – |560	$u_3\,(u_4 0\cdot)$
	$\downarrow$00086|515	
7 (6·4)	– | 45	$u_3\,(u_5 \cdot u_6)$
	$\downarrow$00086|470	
21 (·08)	– 2	$u_3'\,(\cdot 0 u_4)$
	$\downarrow$000|8646|8	
Then by Rule (44), 28 + 8 (·64)	– 3|3	$u_4' + u_4\,(\cdot u_5 u_6)$
Dual number	$\downarrow$0,0,7,8,6,4,3,5,	
	– 35·	7 × 5.

∴ (16) Rule (12), $\downarrow$,(1·00789179) = 7 8 6 0 8 5,

Natural number.	Dual number.	Dual logarithm.
1·00789179	$\downarrow$0,0,7,8,6,4,3,5,	786085,

Ex. 2. *What is the dual number and dual logarithm corresponding to* 1·006?

	100|600|000	
$\downarrow$0,0,5,	100|501|001	$\downarrow^3 u_3,$
	$\downarrow$00|098|999	
5 (90·)	–|450	$u_3\,(u_4 0\cdot)$
	$\downarrow$00|098|549	
5 (8·5)	– 43	$u_3\,(u_5 \cdot u_6)$
	– 1	$u_3'\,(\cdot 0 u_4)$
	$\downarrow$00098505	

To this last result we have to apply the Rule given in article (44).

	$\downarrow$0 0 0 9 8 5 0 5	
$36 + 9(\cdot 84)$	$-$ 4 4	$u_4' + u_4(\cdot u_5 u_6)$
Dual number	$\downarrow$0,0,5,9,8,4,6,1,	

Natural number.	Dual number.	Dual logarithm.
1·006	$\downarrow$0,0,5,9,8,4,6,1,	598221 1,
(12), $u_3(5\cdot)$	2 5 ·	
	5 9 8 2 1 1,	

Ex. 3. *What is the dual number and dual logarithm corresponding to the natural number* 1·00456081 ?

	1 0 0|4 5 6|0 8 1	
$\overset{3}{\downarrow}$ 4,	1 0 0|4 0 0|6 0 0	$\overset{3}{\downarrow} u_3$,
	$\downarrow$0 0|0 5 5|4 8 1	
4(50·)	$-$ |2 0 0	$u_3(u_4 0\cdot)$
	$\downarrow$0 0|0 5 5|2 8 1	
4(5·2)	$-$ 2 1	$u_3(u_5 \cdot u_6)$
	$\downarrow$0 0 0|5 5|2 6 0	
	0	$u_3'(\cdot 0 u_4)$
	$\downarrow$0 0 0|5 5 2 6|0	
$10 + 5(\cdot 52)$	$-$ 1|3	$u^{iv} + u_4(\cdot u_5 u_6)$; (40).
Dual number	$\downarrow$0,0,4,5,5,2,4,7,	

Natural number.	Dual number.	Dual logarithm.
1·00456081	$\downarrow$0,0,4,5,5,2,4,7,	455047,
(12), $u_3(5\cdot)$	2 0 ·	
	4 5 5 0 4 7,	

In this example, given previously, page 45, $u^{iv}(\cdot 0 u_4) = 0$, and $u_3(u_4 u_5 u_6)$ may be subtracted all together, without involving error, as the values of $u_4\ u_5\ u_6$ in this case can be anticipated.

The Author of the present Work communicated examples under Rule (46), page 63, when he first made known Rule (12), see page 16. The original form, in which Rule (12) was delivered, was slightly altered in being analyzed, but the alteration did not involve error; however, the change renders the reversing of the rule difficult; the reverse rule is given in article (13), page 17. The case is otherwise, from changing the original forms when analyzing the examples under Rule (46), as error may be involved; these discrepancies will be discussed towards the end of this chapter.

Ex. 4. *Find the dual number and logarithm corresponding to the natural number* 1·01.

$\overset{3}{\downarrow} 9,$	101|000|000	
	− |903|608	
	↓00|096|392	
9 × 90·	− |810	$u_3(u_4 0\cdot)$
	↓00009 5582	
9(5·5)	− 50	$u_3(u_5 \cdot u_6)$
36(·09)	− 3	$u_3'(\cdot 0 u_4)$
	↓00009 5529	
Then, (40),	− 41	$u_3' + u_4(\cdot u_5 u_6)$

Required dual number ↓0,0,9,9,5,4,8,8,

$$\downarrow, (1\cdot 01) = 995033, \ (16).$$

The natural number 1·01 may be represented by two dual numbers, namely, ↓0,1,0,0,0,0,0,0, and ↓0,0,9,9,5,4,8,8, the dual digits of each being less than 10; other natural numbers may be similarly expressed. $u_3(u_4 u_5 \cdot u_6)$ must be subtracted in two parts, $u_3(u_4 0.)$ and $u_3(u_4 u_5 \cdot u_6)$ and not all together. In the following syntheses these matters will be attended to.

SYNTHESES OF PARTICULAR DEVELOPMENTS. FUNCTIONS AND THEIR INVERSE. OPERATIONS AND THEIR REVERSE.

47. The remainder of this chapter is devoted to matters of importance, which require particular attention.

The dual logarithm of any given number n, divided by 10^8 gives the hyperbolic logarithm of n, to eight places of decimals;

$$\text{that is } \frac{\downarrow,(n)}{10_8} = \log_e n.$$

The young student may be deceived by this coincidence, and imagine that the dual system of logarithms is established by similar processes of reasoning to those used for hyperbolic and common logarithms. That such is not the case may be readily established as follows. Writers on logarithms, with much difficulty and by a series of artifices, show that in the equation

$$r^x = n;$$

x being the logarithm of any given number n, to the base r, that

$$x = \frac{(n-1) - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \frac{1}{4}(n-1)^4 + \dots}{(r-1) - \frac{1}{2}(r-1)_2 + \frac{1}{3}(n-1)^3 - \frac{1}{4}(n-1)^4 + \dots} \quad (Q.)$$

But the expression (Q) cannot be practically applied except in very rare cases. When the denominator is put $= 1$, that is, when

$$(r-1) - \tfrac{1}{2}(r-1)^2 + \tfrac{1}{3}(r-1)^3 - \tfrac{1}{4}(r-1)^4 + \dots = 1,$$

then $\qquad r = 2{\cdot}718281828 \dots$

which, by writers is generally represented by ϵ, and the system is usually termed the hyperbolic system of logarithms.

Let $\qquad (2{\cdot}718281828 \dots)^x = 10;$

then from (Q),

$$x = (10-1) - \tfrac{1}{2}(10-1)^2 + \tfrac{1}{3}(10-1)^3 - \tfrac{1}{4}(10-1)^4 \dots;$$

but to sum this series is practically impossible.

Again, let

$$(2{\cdot}718281828\ldots)^x = 2.$$

Then from (Q),

$$\begin{aligned} x &= (2-1) - \tfrac{1}{2}(2-1)^2 + \tfrac{1}{3}(2-1)^3 - \tfrac{1}{4}(2-1)^4 + \ldots \\ &= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots. \\ &= {\cdot}69314718 \text{ the hyperbolic log. of } 2. \end{aligned}$$

To find the sum of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots$ step by step until we arrive at ·69314718 is a very tedious process.

The hyp. logs. of 1·1; 1·01; 1·001; &c. are more readily found by (Q), for let

$$(2{\cdot}718281828\ldots)^x = 1{\cdot}1,$$

then

$$\begin{aligned} x &= (1{\cdot}1 - 1) - \tfrac{1}{2}(1{\cdot}1 - 1)^2 + \tfrac{1}{3}(1{\cdot}1 - 1)^3 - \tfrac{1}{4}(1{\cdot}1 - 1)^4 + \ldots \\ &= \frac{1}{10} - \tfrac{1}{2}\left(\frac{1}{10}\right)^2 + \tfrac{1}{3}\left(\frac{1}{10}\right)^3 - \tfrac{1}{4}\left(\frac{1}{10}\right)^4 + \ldots \\ &= {\cdot}09531081 \text{ the hyperbolic log. of } 1{\cdot}1 \end{aligned}$$

the denominator of (Q) being $= 1{\cdot}$ when $r = 2{\cdot}71828\ldots.$

Now let

$$(1{\cdot}00000001)^x = 2.$$

then $r = 1{\cdot}00000001$ and $n = 2{\cdot}$ in this case all the terms of the denominator of (Q) may be neglected except

$$r - 1 = {\cdot}00000001 = \frac{1}{10^8}$$

for $\frac{1}{2}(r-1)^2$; $\frac{1}{3}(r-1)^3$; $\frac{1}{4}(r-1)^4$; &c. are very small.

In this latter case (Q) gives

$$x = \frac{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots}{{\cdot}00000001 + \ldots}$$

48. Hence, the value of x in $(2{\cdot}718281828\ldots)^x = 2{\cdot}$ found by (Q), and multiplied by 10^8, is equal to the value of x in $(1{\cdot}00000001)^x = 2{\cdot}$ to eight places of decimals, found also by (Q). The same may be said in applying (Q) to the equations

$$(2{\cdot}718281828\ldots) = 1{\cdot}1 \quad \text{and} \quad (1{\cdot}00000001)^x = 1{\cdot}1$$

$$(2{\cdot}718281828\ldots)^x = 1{\cdot}01 \quad \text{and} \quad (1{\cdot}00000001)^x = 1{\cdot}01$$

&c. &c.

and generally to $\quad \epsilon^x = n \quad$ and $(1{\cdot}00000001)^x = n$,

but, as before observed, the cases in which (Q) is practically applicable, are very few.

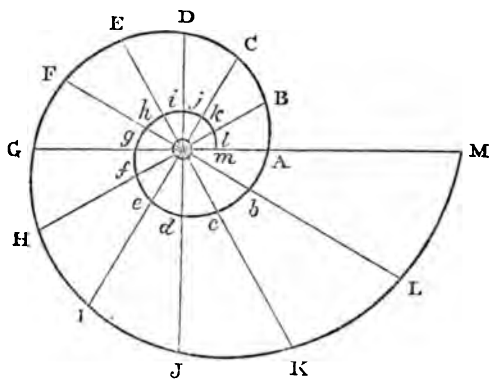
These remarks apply to developments with hyperbolic logarithms given in the Analysis of "Dual Arithmetic, a New Art," pp. 39 to 46. Without care, those developments, with hyperbolic logarithms may give a wrong impression.

Although (Q) indicates that $\frac{\downarrow,(n)}{10^8}$ = the hyperbolic log. of n, true to eight places of decimals for any given number n, yet none of the processes or devices usually employed to apply (Q), to establish it, or to give it a more practical form, in any way resemble the dual system for finding the logarithm of any given number n, to any base r, which is by a direct and extremely simple procedure. The young student will avoid being deceived, by carefully comparing (Q) and "Analysis," pp. 39 to 46, with the correct dual methods of reduction, Chapters I. and IV. of the present work, and "Dual Arithmetic, a New Art," pp. 212 to 214.

The dual system of logarithms furnishes all the advantages of both hyperbolic and common logarithms without retaining any of their defects.

CIRCUMSTANCES UNDER WHICH THE CALCULUS OF DIFFERENCES AND THE DUAL CALCULUS COINCIDE.

49. In the accompanying Figure let each of the angles AOB, BOC, COD, &c. be equal 30°; OA = 1; OB = ↓1,; OC = ↓2,; OD = ↓3,; &c. to OM = ↓12,; and let a logarithmic spiral pass through the points A, B, C, &c. to M. Again, let O*b* = '1↑; O*c* = '2↑; O*d* = '3↑; &c. to O*m* = '12↑; and let a logarithmic spiral pass through the points A, *b*, *c*, *d*, &c. The radius vector ↓2,5, falling half-way between C and D falls beyond the curve,



for the length of a radius vector drawn to the curve half-way between C and D, that is forming an angle of 75° with OA, is equal ↓2,4,7,8,5,7,2,7, true to eight places of decimals.

In the descending branch the radius vector '4'5↑ does not fall on the curve half-way between *e* and *f*, or which is the same thing, the radius vector '4'5↓ does not make an angle of 135° with OA.

'4'5'2'4'2'7'5'6↑ is the length of the radius vector to the curve in middle between *e* and *f*. It is easily observed that the

calculus of differences will only apply to ten consecutive digits of the same rank when true, and not approximate values, are required. A radius vector whose length is $\downarrow 2,5,$ makes an angle of $75° \; 39' \; 35''·7$ with OA, and not an angle of $75°$; and a radius vector of $'4'5 \uparrow$ makes an angle of $135° \; 43' \; 58·4''$, and not an angle of $135°$.

50. The operative numbers will not apply to all natural numbers that correspond to consecutive dual numbers. One or two of the numerous examples that might be selected will illustrate this matter.

$$\begin{array}{l}
\downarrow 0,9,9, = 1·10356790 \\
\downarrow 1,0,0, = 1·10000000 \\
\downarrow 1,0,1, = 1·10110000 \\
\hline
\downarrow 1,9,9, = 1·21392468 \\
\downarrow 2,0,0, = 1·21000000 \\
\downarrow 2,0,1, = 1·21121000 \\
\hline
\quad \&c. \qquad \&c.
\end{array}$$

One example from the descending branch will be sufficient.

$$\begin{array}{l}
·65998566 = '3'9'9 \uparrow \\
·65610000 = '4'0'0 \uparrow \\
·65544390 = '4'0'1 \uparrow \\
\quad \&c. \qquad \&c.
\end{array}$$

Although the operative numbers do not conduct us from the value of $\downarrow 1,9,9,$ to the value of $\downarrow 2,0,0,$ &c. yet they will apply in passing from $\downarrow 5,5,0,9,$ to $\downarrow 5,5,1,9,$; $\downarrow 5,5,2,9,$ &c.

$$\begin{array}{lr}
\downarrow 5,5,0,9, = & 1·69418619 \\
 & 169419 \\
\hline
\downarrow 5,5,1,9, = & 1·69588038 \\
 & 169588 \\
\hline
\downarrow 5,5,2,9, = & 1·69757626 \\
 & 169758 \\
\hline
\downarrow 5,5,3,9, = & 1·69927384 \\
\&c. & \&c.
\end{array}$$

otherwise

$$\begin{array}{r|r|r}
169 & 418 & 619 \\
 & 508 & 256 \\
 & & 508 \\
\hline
\multicolumn{3}{r}{1·69927383}
\end{array}$$

See (37), page 58.

A logarithmic spiral may be made to pass through the radii vectors $\downarrow 5{,}5{,}0{,}9{,}$; $\downarrow 5{,}5{,}1{,}9{,}$; $\downarrow 5{,}5{,}2{,}9{,}$; &c. with the angular distances between every consecutive pair equal.

The operative numbers are employed both in the calculus of differences, and in the dual calculus, but under different restrictions; for example,

$$\begin{aligned}\downarrow 5{,}5{,}9{,}7{,} &= 1{\cdot}70915315\\ \downarrow 5{,}5{,}9{,}8{,} &= 1{\cdot}70932407\\ \downarrow 5{,}5{,}9{,}9{,} &= 1{\cdot}70949500\\ \downarrow 5{,}6{,}0{,}1{,} &= 1{\cdot}70975978\end{aligned}$$

The calculus of differences will show that

$$\downarrow 5{,}5{,}9{,}9{,}5{,}4{,}8{,}8{,} = 1{\cdot}70958882$$

but

$$\downarrow 5{,}6{,}0{,}0{,}0{,}0{,}0{,}0{,} = 1{\cdot}70958882$$

The equality here established is only correct as far as eight places of decimals, for $\downarrow 5{,}6{,}0{,}0{,}0{,}0{,}0{,}0{,} = 1{\cdot}709588817744 41651$ exactly. $\downarrow 5{,}5{,}9{,}9{,}5{,}4{,}8{,}8{,}$ has an exact value also, (32) page 47, the first nine figures of which do not differ a unit from 170958882. The calculus of differences fails to determine the exact value of $\downarrow 5{,}6{,}$ by the consecutive differences above employed.

As we proceed, other comparisons and parallel developments will be instituted, and it will be finally demonstrated that the calculus of differences, when properly restricted, becomes a branch of the dual calculus. When the Analysis was being drawn up, the Author of the present Work introduced the calculus of differences to show how the operative numbers might be derived without reference to the binomial theorem, and also to show how to construct a table of ascending dual numbers with their corresponding natural numbers, by common addition, and independent of the operative numbers; the alterations made in analyzing the first communication, rendered the explanations, entered into here, necessary.

A COMBINATION OF PARTICULAR FACTORS THAT MAY MISLEAD WHEN MADE TO ASSUME THE FORM OF AN ASCENDING DUAL NUMBER.

51. The factors of the imitative arrangement are

$$\begin{array}{llllll} 1\cdot9 & 1\cdot8 & 1\cdot7 & \ldots\ldots\ldots & 1\cdot1 & \\ 1\cdot09 & 1\cdot08 & 1\cdot07 & \ldots\ldots\ldots & 1\cdot01 & (F) \\ 1\cdot009 & 1\cdot008 & 1\cdot007 & \ldots\ldots\ldots & 1\cdot001 & \\ & \&c. & & & \&c. & \end{array}$$

It is easily shown that

$$1\cdot8389270996 =$$
$$(1\cdot7)\,(1\cdot08)\,(1\cdot001)\,(1\cdot0005)\,(1\cdot00009)\,(1\cdot000003)\,(1\cdot0000006)\,(1\cdot00000004).$$

The factors 1·1, 1·01, 1·001, &c. are seldom incorporated in such products. 1·3 is put for $(1\cdot1)^3$; 1·9 for $(1\cdot1)^9$; 1·06 for $(1\cdot01)^6$; 1·007 for $(1\cdot001)^7$; &c. These counterfeit factors are rigid and inflexible, and have no branch to imitate descending dual digits.

The log of 1·1 being given, the log of 1·9, 1·7, 1·6, &c. cannot be readily found; while in the dual system numbers are expressed by indices, and not by coefficients, and are very flexible, besides, when log 1·1 is known, log $(1\cdot1)^9$, log $(1\cdot1)^7$, log $(1\cdot1)^6$ &c. are easily found.

52. An overrated method is given in the Analysis, pp. 61 to 72, "Dual Arithmetic, a New Art," to find dual logarithms by limited tables of the logarithms of the factors (F); this method may be applied to other systems of logarithms, but not without limited tables which the method cannot supply in any case.

However, the logarithms of the factors (F) can only be independently calculated by the dual method. For example, the dual or any other logarithm of 1·8389270996 may be found by adding together the logarithms of the factors (1·7), (1·08), (1·001), &c. taken from tables previously prepared.

53. The factors (F) have also been employed to approximate to the roots of particular equations; a root so determined might be put under a form to imitate a dual result, but a slight inspection renders the difference apparent, even to those who merely understand the application of the ascending branch of the dual calculus to find unknown quantities under a variety of dual forms, which subject is fully discussed in the next Chapter.

OPERATIONS AND THEIR REVERSE.

54. It is a very important feature of the dual calculus, especially in finding the roots of equations, that inverse dual functions are not only expressed compactly but also readily determined; and in most cases, dual operations are readily reversed. Before the introduction of dual arithmetic but few elementary functions possessed these useful properties. A direct rule should be so framed that the reverse one may be easily deduced when required; these important features have not been dwelt upon, indeed, they have been much neglected.

The rule article (12), page 15, is taken from the expression

$$\downarrow, u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, = u_1 u_2 u_3 u_4 u_5 u_6 u_7 u_8 - 5\,(u_1 0 u_2 0 u_3 0) \\ + 31018 u_1 + 33 u_2;\ (16).$$

But in reversing the Rule, $5\,(u_1 0 u_2 0 u_3 0)$ has to be added, and $31018 u_1$ and $33 u_2$ to be subtracted. Hence the expression for the reverse Rule given in article (13) page 17 must be put under the form

$$\begin{aligned} u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, &- 5\,(u_1 00000) + 31018 u_1 \\ &- 5\,(u_2 000) \quad + 33 u_2 \\ &- 5\,(u_3 0) \end{aligned}$$

Since $5\,(u_1 0 u_2 0 u_3 0) = 5\,(u_1 00000) + 5\,(u_2 000) + 5\,(u_3 0)$.

This modification is necessary, as the values of u_1 u_2 have often to be anticipated in applying the reverse Rule.

For example, let it be required to reduce the dual logarithm 29950000, to a dual number.

Dual log	29950000,	
5($u_1$00000)	+ 1500000	5(300000)
	31450000	
31018u_1	− 93054	3(31018)
	31356946	
5($u_2$000)	+ 5000	5(1000)
	31361946	
33u_2	− 33	1(33)
	31361913	
5($u_3$0)	+ 150	5(30)
Dual number	$\downarrow$ 3,1,3,6,2,0,6,3,	see (16).

Again, let it be required to reduce $\downarrow$ 3,1,3,6,2,0,6,3 to a dual logarithm.

Dual number	$\downarrow$ 3,1,3,6,2,0,6,3,	
5($u_1$0$u_2$0$u_3$0)	− 1505150	5(301030)
	29856913	
u_1(31018)	+ 93054	3(31018)
u_2(33)	+ 33	1(33)
Dual logarithm	29950000,	see (12).

The Author regrets having allowed the original form under which he communicated this Rule (12), to be altered when being analyzed, for, as before remarked, page 66, although the change did not involve error, yet it renders the reversing of the rule difficult.

ORIGINAL FORM.

Article (12), *page* (15).

55. *Rule*:—From the dual number of eight digits taken as a common number, subtract five times the first three digits, supposing a cipher placed after each, add 31018 times the first digit, and 33 times the second, the amount is the dual logarithm. In most cases, this reduction may be effected in one operation.

Thus,	$+\ 33$	$= 1(33)$
	$+\ 93054$	$= 3(31018)$
	↓ 3,1,3,6,2,0,6,3,	Dual number.
	$-\ 150515^{\cdot}$	$= 5(301030)$
Dual logarithm	29950000,	

It is advisable that the student, before proceeding further, should thoroughly understand the criticisms instituted from Article (41) to Article (55).

Functions and their inverse, operations and their reverse, will be discussed in a general manner when the ascending and descending branches are combined, for not until then can the great power of the dual calculus be applied. In Chapter IV. the descending branch will be treated of systematically, and in detail.

CHAPTER III.

ASCENDING DUAL DEVELOPMENTS APPLIED TO DETERMINE THE VALUES OF UNKNOWN QUANTITIES UNDER A VARIETY OF DUAL FORMS.

56. In this Chapter we do not discuss the theory of equations, nor establish any abstract criteria respecting the nature of the roots of equations, but apply a system that will determine the values of unknown quantities under a variety of ascending dual forms, and that too without being obliged to keep within very narrow boundaries. In fact, we first propose to show the power and scope of the machinery to be put in motion, and afterwards to restrict the operations of the whole machine to concise and convenient limits.

Examples.

Ex. 1. *Given* $276{\cdot}593124x = 7634{\cdot}83528$, *to find the 7th root and dual logarithm of* x.

$$x = \frac{7634{\cdot}83528}{276\ 593124} = \frac{2^2 10^3(1{\cdot}90870882)}{(2)10^2(1{\cdot}38296562)} = (2)(10) \downarrow 3,3,6,4,1,8,2,9,$$

$$(2)(10) \downarrow 3,3,6,4,1,8,2,9, = \overset{8}{\downarrow} 331792909,$$

$$\therefore \quad \downarrow, (x) = 331792909, \qquad \text{Article (16).}$$

$$\frac{\downarrow, (x)}{7} = 47398987, = \downarrow, 4,9,3,1,9,7,6,8,$$

$\downarrow 4,9,3,1,9,7,6,8, = 1{\cdot}60639071$, the 7th root of x.

INVESTIGATION.

57. Put 1·38296562 = A and 1·90870882 = B;

Now $\frac{B}{A}$ is greater than 1·33, but less than 1·46, hence the first dual digit is $\downarrow 3,$. Or, $\frac{B-A}{A}$ is greater than ·33, but less than ·46, which also shows the first dual digit to be $\downarrow 3,$; the three first figures of A and B have only to be inspected to arrive at this result. Should a digit be taken too great, the work may be continued by making the succeeding digit negative.

```
B = 1 9 0 8 7 0 8 8 2
    ─────────────────
A = 1|3|8|2|9|6|5 6 2
     |4|1|4|8|8|9 6 9    ↓3,
       |4|1|4|8|8 9 7
         |1|3|8|2 9 7
    ─────────────────
```
$A \underset{\downarrow}{+} u_1 A = A(1 \underset{\downarrow}{+} u_1) = 1\,8\,4\,0\,7\,2\,7\,2\,5$ put $= b$.

Now $\frac{B}{b}$ is greater than 1·030, but less than 1·040, or $B-b$ divided by b is greater than ·030, but less than ·040, hence the next dual digit is $\downarrow 0,3,$

```
A(1 ∔ u₁) = 18|4 0|7 2|72|5 .  put = b
               |5 5|2 2|18|2 .
                   |5 5|22|2 .  ↓0,3,
                       |18|4 .
```
$A\{(1 \underset{\downarrow}{+} u_1) \underset{\downarrow}{+} [u_2\} = b(1 \underset{\downarrow}{+} u_2) =$ 18 9|6 5 0|31 3 put $= c$

```
   1|1 3 7|90 2
        2|84 5   ↓0,0,6,
             4
```
$A\{(1 \underset{\downarrow}{+} u_1) \underset{\downarrow}{+} [u_2 \underset{\downarrow}{+} [u_3\} = c(1 \underset{\downarrow}{+} u_3) =$ 1907|9 1 06|4 put $= d$

```
     |7 6 31|6
           1|1   ↓0,0,0,4,
─────────────
19086|7 39 1     put = e
```

$$
\begin{array}{r|l}
\underset{\cdot\ \cdot\ \dagger}{19086} & \underline{7391} \quad \text{put} = e \\
 & 3491 = \mathrm{B} - e \\
 & \underline{1909} = \overset{5}{\downarrow} 1, \\
 & 1582 \\
 & \underline{1527} = \overset{6}{\downarrow} 8, \\
 & \quad 55 \\
 & \quad \underline{38} = \overset{7}{\downarrow} 2, \\
 & \quad 17 \\
 & \quad \underline{17} = \overset{8}{\downarrow} 9, \\
\end{array}
$$

$$
\begin{aligned}
\therefore \quad & \mathrm{A}\{(1 \dagger u_1) \dagger [u_2 \dagger [u_3 \dagger [u_4 \dagger [u_5 \dagger [u_6 \dagger [u_7 \dagger [u_8\} \\
& = \mathrm{A} \downarrow u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, = \mathrm{B}
\end{aligned}
$$

Or $$\mathrm{A} \downarrow 3,3,6,4,1,8,2,9 = \mathrm{B}$$

$$\therefore \quad \frac{\mathrm{B}}{\mathrm{A}} = \downarrow 3,3,6,4,1,8,2,9,$$

$$
\begin{aligned}
& b\{(1 \dagger u_2) \dagger [u_3 \dagger \ldots\ldots \dagger [u_8\} = \mathrm{B}; \\
& c\{(1 \dagger u_3) \dagger [u_4 \dagger \ldots\ldots \dagger [u_8\} = \mathrm{B}; \\
& d\{(1 \dagger u_4) \dagger [u_5 \dagger \ldots\ldots \dagger [u_8\} = \mathrm{B}; \\
& \quad \text{\&c.} \qquad\qquad\qquad \text{\&c.}
\end{aligned}
$$

Since,

$$(1 \dagger u_1) \dagger [u_2 \dagger [u_3 \dagger \ldots\ldots = \frac{\mathrm{B}}{\mathrm{A}}$$

$$\therefore \quad 1 \dagger u_1 = \frac{\mathrm{B}}{\mathrm{A}} \text{ minus } \dagger [u_2 \dagger [u_3 \dagger \ldots \qquad (1).$$

In finding u_1 we have not to estimate the value of $\dagger [u_2 \dagger [u_3 \dagger \ldots$ all we want to know respecting it is that it will not increase the value of u_1 a unit.

Take $\frac{B}{A} = 1{\cdot}79\ldots$, as another example to illustrate the notation; then $u_1 = \downarrow 6$, for $(1{\cdot}1)^6 = 1{\cdot}771561$; and $(1{\cdot}1)^7 = \downarrow 7, = 1{\cdot}9487171$.

Hence, between $\downarrow 6, = 1{\cdot}771561$ and $\downarrow 7, = 1{\cdot}9487171$ $\downdownarrows[u_2 \dotplus [u_3 \dotplus \ldots$ has a range of values between 0, and $\cdot 1771561$, which is the $\frac{1}{10}$ of the natural number corresponding to the lesser dual digit. Neglecting $\dotplus [u_2 \dotplus [u_3 \dotplus \ldots$ in (1) and putting

$$A \dotplus A u_1 = B$$

$$\therefore \quad \dotplus u_1 = \frac{B - A}{A}.$$

Then it is clear, if $\frac{B-A}{A} = \cdot 79$ or $\cdot 87$ or any number up to $\cdot 95$, $u = \downarrow 6$; but if $\frac{B-A}{A} = \cdot 34$ or $\cdot 42$ or any number between $\cdot 331$ and $\cdot 4641$, $u_1 = \downarrow 3$,

Again, because

$$(1 \dotplus u_2) \dotplus [u_3 \dotplus [u_4 \dotplus \ldots = \frac{B}{b},$$

$$\therefore \quad 1 \dotplus u_2 = \frac{B}{b} \text{ minus } \dotplus [u_3 \dotplus [u_4 \dotplus \ldots \quad (2)$$

u_2 may be so chosen, that the rejection of $\dotplus [u_3 \dotplus [u_4 \dotplus \ldots$ will not decrease the value of u_2 a unit, or render u_3 negative. If $\frac{B}{b} = 1{\cdot}032$, then $u_2 = \downarrow 0{,}3$, and u_2 may be taken $= \downarrow 0{,}3$, for all values between $1{\cdot}030301$ and $1{\cdot}04060401$; hence, between $\downarrow 0{,}3$, and $\downarrow 0{,}4$, ; $\dotplus [u_3 \dotplus [u_4 \dotplus \ldots$ has a range of values between 0 and $\cdot 01030301$, which is the $\frac{1}{100}$th part of $1{\cdot}030301$. Neglecting $\dotplus [u_3 \dotplus [u_4 \dotplus \ldots$ in (2), then

$$b(1 \dotplus u_2) = b \dotplus u_2 b = B$$

$$\therefore \quad \dotplus u_2 = \frac{B - b}{b}.$$

If $\dfrac{B-b}{b} = \cdot 032$, then $u_2 = \downarrow 0{,}3$, if $\dfrac{B-b}{b} = \cdot 074\ldots$, $u_2 = \downarrow 0{,}7$, for $\cdot 074\ldots$ is greater than $\cdot 07213535210701$ and less than $\cdot 08285670562808 01$. In a similar manner u_3, may be found from

$$1 \downarrow u_3 = \frac{B}{c},$$

or from
$$\downarrow u_3 = \frac{B-c}{c},$$

reserving for further consideration, the surplus, represented by $\downarrow [u_4 \downarrow [u_5 \downarrow$. The process being continued, the dual number is found under the simplest form, when made up of ascending dual digits.

It may be necessary to observe, that as many places of decimals are taken as the required dual number is to have digits

Ex. 2. Given $7634{\cdot}83528x = 276{\cdot}593124$, *to find* x, *its fifth root, and dual logarithm.*

$$x = \frac{276{\cdot}593124}{7634{\cdot}83528} = \frac{(10)^2(2{\cdot}76593124)}{(2)^2(10)^3(1{\cdot}90870882)} = \tfrac{1}{40} \downarrow 3,8,5,4,1,9,7,2,$$

or,
$$x = \cdot 0362277787 5.$$

$$\downarrow, (4) = 138629436,$$
$$\downarrow, (10) = 230258509,$$
$$\downarrow, (40) = 368887945,$$

$$\therefore \quad \downarrow, (\tfrac{1}{40}) = '368887945 \quad \text{See Article (17)}$$
$$\downarrow, 3,8,5,4,1,9,7,2, = 37095040,$$
$$5\;)\;'331792905 \quad \text{dual log of } x \text{ or } \downarrow, (x)$$
$$'66358581$$

$$'66358581 \uparrow = \tfrac{1}{2} \downarrow 2956137, = \tfrac{1}{2} \downarrow 0,2,9,6,6,5,1,9, = \cdot 51500129;$$

$$\therefore \quad \text{the fifth root of } x = \cdot 51500129.$$

These and similar results may be found by other dual methods of operating, and under a variety of dual forms; however, the importance of the particular treatment here employed, will be presently seen when we come to find the roots of complex equations.

Details of the work of the last Example.

2 7 6 5 9 3 1 2 4

↓3,
1 | 9 | 0 | 8 | 7 | 0 | 8 | 8 | 2
| 5 | 7 | 2 | 6 | 1 | 2 | 6 | 5
| 5 | 7 | 2 | 6 | 1 | 2 | 7
| 1 | 9 | 0 | 8 | 7 | 1

2765
1908
1908) 857 (1·45
763
94
95

Therefore
↓u_1 = ↓3,
because 1·45 is less than
1·4641 = ↓4,
but greater than
1·331 = ↓3,

$\overset{2}{↓}$8,
2 5 | 4 0 | 4 9 | 1 4 | 5 .
2 | 0 3 | 2 3 | 9 3 | 2 .
7 | 1 1 | 3 3 | 8 .
| 1 4 | 2 2 | 7 .
| 1 7 | 8 .
| 1 .

$\overset{3}{↓}$5,
2 7 5 | 0 9 8 | 8 2 1
1 | 3 7 5 | 4 9 4
2 | 7 5 1
3

27659
25404
25404) 2255 (1·088 . .
2032
223
203

Therefore
↓u_2 = ↓0,8,
because 1·088 is
less than
1·0936 . . = ↓0,9,
but greater than
1·082856 . . = ↓0,8,

$\overset{4}{↓}$4,
2 7 6 4 | 7 7 0 6 | 1
1 | 1 0 5 9 | 1
1 | 7

2 7 6 5 8 | 7 6 6 9
| 5 4 5 5
2 7 6 6
2 6 8 9
2 4 8 9
2 0 0
1 9 4
6

276593
275098
275098) 1495 (1·0054 . .
1376
119

Therefore
↓0,0,u_3 = ↓0,0,5,
because 1·0054 is
between
↓0,0,5, = 1·0510 . .
and
↓0,0,6, = 1·0615 . . .

58. Given $ax^2 + bx = c$, to find the value of x.

Substitute r for x, so that it may be possible to represent x under the form

$$r \downarrow u_1, u_2, u_3, \ldots .$$

the nearer r approaches the value of x, the less will be the dual number $\downarrow u_1, u_2, u_3, \ldots$ which has an extensive range of forms and values in each particular case. Criteria to determine the range and convenient limits of r, in equations of all degrees, will be treated of in another place.

Let $$ar^2 + br = c_1,$$

and suppose the first dual digit to be $\downarrow u_1$, then substituting $r \downarrow u_1$, for x,

$$ar^2 \downarrow 2u_1, + br \downarrow u_1, = c;$$

$$\therefore \quad ar^2(1 \dotplus 2u_1) + br(1 \dotplus u_1) = c;$$

$$ar^2 + br \dotplus 2ar^2u_1 \dotplus bru_1 = c;$$

$$c_1 \dotplus 2ar^2u_1 \dotplus bru_1 = c;$$

$$\therefore \quad u_1 = \frac{c - c_1}{\dotplus 2ar^2 \dotplus br}.$$

u_2, u_3, &c. may be found in a similar manner.

Since u_1, u_2, &c. are whole numbers, positive or negative, $\dotplus 2ar^2 \dotplus br$ in most cases may be treated as $+ 2ar^2 + br$.

It may be necessary to state that but little practical inconvenienace can rise at any time in making u_n, a unit greater or less than the proper dual digit belonging to the nth position of the required dual number presented under its simplest form; for if u_n, be taken too great, then by making the following dual digit negative, the process may be continued without retraction, interruption, or error. The same may be said if u_n, be taken too small, with this difference, that the succeeding dual digit will be greater than 9_1.

Ex. 3. *Given* $357{\cdot}836528x^2 - 573{\cdot}456388x = 8107{\cdot}37676$, *to find both values of* x, *and the 7th root and dual logarithm of the lesser value.*

If we examine $(3{\cdot}5\ldots.)x^2 - (5{\cdot}7\ldots)x = 81{\cdot}0\ldots$ (the given equation divided by 100), it appears that $3\downarrow u_1$; $4\downarrow u_1$; $5\downarrow u_1$, may be substituted for x; $6\downarrow u_1$ also may be substituted for x, but then u_1 becomes negative, and the operation is not as easily performed as when $5\downarrow u_1$, is put for x. Then putting $x = 5\downarrow u_1$, the given equation becomes

$$8945{\cdot}9132\downarrow(2u_1), - 2867{\cdot}28194\downarrow u_1 = 8107{\cdot}37676,$$

which has to be compared with the general equation (58)

$$ar^2\downarrow 2u_1 - br\downarrow u_1 = c$$

$$\begin{array}{lrlrl}
 & +178 & \text{twice} & +89\ldots & (a)r^2 \\
 & -\ \ 28 & \text{once} & -28\ldots & (b)r \\
\hline
(+2ar^2 - br) & +150 & & +61\ldots & (c_1) \\
 & & & +81\ldots & (c) \\
\hline
 & & 150) & +20\ldots & (c - c_1) \\
 & & & 15\ldots & (\downarrow 1, = \downarrow u_1
\end{array}$$

Substituting $r\downarrow u_1, u_2$, for x the given equation becomes

$$10824{\cdot}5549\downarrow 2u_2 - 3154{\cdot}01013\downarrow n_2 = 8107{\cdot}37676$$

$$\begin{array}{rlrl}
+2164 & \text{twice} & +10824{\cdot}5549 & \\
-\ 310 & \text{once} & -\ \ 3154{\cdot}01013 & \\
\hline
-\dot{1}\dot{8}54) & & +\ \ 767.\ldots.. & \\
 & & +\ \ 810.\ldots.. & \\
\hline
 & & 43 & \\
 & & 37 & (\downarrow 0{,}2, = \downarrow u_2,
\end{array}$$

The next step furnishes the equation

$$11264{\cdot}0752 \downarrow 2u_3 - 3217{\cdot}40573 \downarrow u_3 = 8107{\cdot}37676$$

$$\begin{array}{rcl}
+\ 22528 & \text{twice} & +\ 11264\ldots \\
-\ \ 3217 & \text{once} & -\ \ 3217\ldots \\
\hline
19311) & & +\ \ 8047\ldots \\
\ldots & & +\ \ 8107\ldots \\
\cline{3-3}
 & & 60 \quad (\downarrow 0,0,3, = \downarrow u_3 \\
 & & 58
\end{array}$$

The next step furnishes the equation

$$11331{\cdot}8288 \downarrow 2u_4 - 3227{\cdot}06760 \downarrow u_4 = 8107{\cdot}37676$$

$$\begin{array}{rclr}
+\ 22663 & \text{twice} & +\ 11331{\cdot}8288 & \\
-\ \ 3227 & \text{once} & -\ \ 3227{\cdot}06760 & \\
\hline
19436 & & +\ \ 8104{\cdot}76120 & \text{take} \\
\ldots\ldots & & +\ \ 8107{\cdot}37676 & \text{from} \\
\cline{3-3}
 & & 2\ 61556 & \\
 & & 1\ 9436 & (\downarrow 0,0,0,1,3,4,5,7, \\
\cline{3-3}
 & & 6719 & \\
 & & 5831 & \\
\cline{3-3}
 & & 888 & \\
 & & 777 & \\
\cline{3-3}
 & & 112 & \\
 & & 97 & \\
\cline{3-3}
 & & 15 & \\
 & & 14 & \\
\cline{3-3}
\end{array}$$

$$\therefore\ x = \downarrow 1,2,3,1,3,4,5,7, = 5{\cdot}62915577$$

Coefficient of second term with its sign changed

$$= \frac{573{\cdot}456388}{357{\cdot}836528} = 1{\cdot}60256526$$

$$\begin{array}{rr}
\text{From} & 1{\cdot}60256526 \\
\text{Take} & 5{\cdot}62915577 \\
\hline
\text{Also}\quad x = & -\ 4{\cdot}02659051
\end{array}$$

$5 \downarrow 1,2,3,1,3,4,5,7, = \downarrow 172778181,$ the dual logarithm of x.

x may also be found under the form, $x = 4 \downarrow 3,5,5,8,0,7,7,6,$

$7) \downarrow 172778181,$

Seventh root $\downarrow\ 24682597, = \downarrow 2,5,6,4,5,\dot{6},9,6, = 1{\cdot}27995632$

59. Given $ax^3 + bx^2 + cx = d$, find a.

Let r be taken so that the required root may be found under the form $r \downarrow u_1, u_2, u_3, \ldots.$ As before remarked, r has a great range of values, but it is evident that the nearer r is taken to the value of x, the less will be the affixed dual number $\downarrow u_1, u_2, u_3, \ldots\ldots$ which may be made to assume a variety of forms in each particular case.

Substitute $r \downarrow u_1$ for x in the given equation, then

$$ar^3 \downarrow 3u_1 + br^2 \downarrow 2u_1 + cr \downarrow u_1 = d,$$

$$\therefore\quad ar^3(1 \dotplus 3u_1) + br^2(1 \dotplus 2u_1) + cr(1 \dotplus u_1) = d,$$

$$\therefore\quad ar^3 + br^2 + cr \dotplus 3u_1ar^3 \dotplus 2u_1br^2 \dotplus u_1cr = d,$$

$$\therefore\quad u_1 = \frac{d - d_1}{\dotplus 3ar^3 \dotplus 2br^2 \dotplus cr},$$

putting d_1 for $ar^3 + br^2 + cr$. The dual digit u_1 may be obtained by employing $+ 3ar^3 + 2br^2 + cr$ as a division instead of $\dotplus 3ar^3 \dotplus 2br^2 \dotplus cr$. To find x, put $ar^3 \downarrow 3u_1 = a_1$; $br^2 \downarrow 2u_1 = b_1$; $cr \downarrow u_1 = c_1$; and substituting $r \downarrow u_1, u_2,$ for x, the given equation becomes

$$a_1 \downarrow 3u_2 + b_1 \downarrow 2u_2 + c_1 \downarrow u_2 = d,$$

or, $$a_1(1 \dotplus 3u_2) + b_1(1 \dotplus 2u_2) + c_1(1 \dotplus u_2) = d$$

$$\therefore\quad u_2 = \frac{d - d_2}{\dotplus 3a_1 \dotplus 2b_1 \dotplus c_1};$$

d_2 being put for $a_1 + b_1 + c_1$. As in the case of u_1, the value of u_2 may be found by employing $+ 3a_1 + 2b_1 + c_1$ as a divisor

instead of $\dagger 3a_1 \dagger 2b_1 \dagger c_1$. Again putting $a_1 \downarrow 3u_2 = a_2$; $b_1 \downarrow 2u_1 = b_2$; $c_1 \downarrow u_2 = c_2$; then by substituting $r \downarrow u_1' \, u_2' \, u_3'$ for x, the given equation becomes

$$a_2 \downarrow 3u_3 + b_2 \downarrow 2u_3 \dagger c_2 \downarrow u_3 = d,$$

$$\therefore \quad u_3 = \frac{d - d_3}{\dagger 3a_2 \dagger 2b_2 \dagger c_2};$$

By continuing the process and extending our notation

$$u_4 = \frac{d - d_4}{\dagger 3a_3 \dagger 2b_3 \dagger c_3}; \qquad u_5 = \frac{d - d_5}{\dagger 3a_4 \dagger 2b_4 \dagger c_4}; \text{ \&c.}$$

60. In a general equation of the fourth degree,

$$ax^4 + bx^3 + cx^2 + dx = e,$$

let two or all the roots be real, then as in the last case.

$$u_1 = \frac{e - e_1}{\dagger 4ar^4 \dagger 3br^3 \dagger 2cr^2 \dagger dr};$$

$$u_2 = \frac{e - e_2}{\dagger 4a_1 \dagger 3b_1 \dagger 2c_1 \dagger d_1};$$

$$u_3 = \frac{e - e_3}{\dagger 4a_2 \dagger 3b_2 \dagger 2c_2 \dagger d_2}$$

&c. &c.

61. If $r \downarrow u_1, u_2, u_3, \ldots$ be a root of the equation

$$ax^5 + bx^4 + cx^3 + dx^2 + ex = f,$$

then u_1, u_2, u_3, &c. may be found from

$$u_1 = \frac{f - f_1}{\dagger 5ar^5 \dagger 4br^4 \dagger 3cr^3 \dagger 2dr^2 \dagger er}$$

$$u_2 = \frac{f - f_2}{\downarrow 5a_1 \downarrow 4b_1 \downarrow 3c_1 \downarrow 2d_1 \downarrow e_1};$$

$$u_3 = \frac{f - f_3}{\downarrow 5a_2 \downarrow 4b_2 \downarrow 3c_2 \downarrow 2d_2 \downarrow e_2};$$

&c. &c.

Other equations may be treated in the same general manner.

Ex. 4. *Given* $34{\cdot}56789x^5 - 2345{\cdot}678x^4 - 123{\cdot}4567x^3 + 456{\cdot}7891x^2 + 56789{\cdot}12x = -415978976{\cdot}065$ *to find a value of x, true to nine places of decimals.*

It requires but little observation to see that a value of x lies between 0 and 100, and on closer inspection it will be found that a value lies between 10 and 30. Then 20 being put for r, and $r \downarrow u_1$ being substituted for x, u_1 may be obtained from

$$u_1 = \frac{f - f_1}{\downarrow 5ar^5 \downarrow 4br^4 \downarrow 3cr^3 \downarrow 2dr^2 \downarrow er};$$

$ar^5 = a_1$	$br^4 = b_1$	$cr^3 = c_1$	$dr^2 = d_1$	$er = e_1$
$+110617248{\cdot}$	$-375308480{\cdot}$	$-987653{\cdot}6$	$+182715{\cdot}64$	$+1135782{\cdot}4$

+ 550	5 times	+ 110	
− 1500	4 times	− 375	
950	3 times	−	
	2 times	+	
	1 time	+ 1	
		− 264	(f_1) take
		− 415	(f) from
− 950)		− 151	$(+\downarrow 1, = \downarrow u_1,$

$a_1 \downarrow 5u_1 = a_2$	$b_1 \downarrow 4u_1 = b_2$	$c_1 \downarrow 3u_1 = c_2$	$d_1 \downarrow 2u_1 = d_2$
$+ 178150185{\cdot}$	$- 549489146{\cdot}$	$- 1314566{\cdot}942$	$+ 221085{\cdot}924$

$e_1 \downarrow u_1, = e_2$
$+ 1249360{\cdot}64$

+ 8905	5 times	+ 1781	
− 21976	4 times	− 5494	
− 39	3 times	− 13	
+ 4	2 times	+ 2	
+ 12	1 time	+ 12	
− 1309)		− 3712	(f_2) take
		− 4159	(f) from
		− 447	(+ ↓ 0,3, = ↓ u_2
		− 393	

$a_2 \downarrow 5u_2 = a_3$ $\quad b_2 \downarrow 4u_2 = b_3$ $\quad c_2 \downarrow 3u_2 = c_3$ $\quad d_2 \downarrow 2u_2 = d_3$

+ 206826835· − 619178123· − 1437722·51 + 234687·163

$e_2 \downarrow u_2 = e_3$

+ 1287217·52

+ 103410	5 times	+ 20682	
− 247668	4 times	− 61917	
− 429	3 times	− 143	
+ 46	2 times	+ 23	
+ 128	1 time	+ 128	
− 144513)		− 41227	(f_3) take
		− 41597	(f) from
		− 370	(+ ↓ 0,0,2,5, = + ↓ $u_3 u_4$,
		− 289	
		81	

↓ 0,0,2,5,0,0,0,0, = ↓ 249900,

square = ↓ 499800, = ↓ 0,0,5,0,0,0,5,0,

cube = ↓ 749700, = ↓ 0,0,7,5,0,0,5,0,

4th = ↓ 999600, = ↓ 0,1,0,0,4,5,6,7,

5th = ↓ 1254500, = ↓ 0,1,2,5,4,4,6,7,

$a_8 \downarrow 0,1,2,5,4,4,6,7,$ $\quad$ $b_8 \downarrow 0,1,0,0,4,5,6,7,$ $\quad$ $c_8 \downarrow 0,0,7,5,0,0,5,0,$
$+ 209427135\cdot$ $\quad$ $- 625398465$ $\quad$ $- 1448548\cdot10$

$d_8 \downarrow 0,0,5,0,0,0,5,0,$ $\quad$ $e_8 \downarrow 0,0,2,5,0,0,0,0,$
$+ 235863\cdot066$ $\quad$ $+ 1290438\cdot27$

+ 1047135	5 times	+ 209427135·	
− 2501593	4 times	− 625398465·	
− 4345	3 times	− 1448548·10	
+ 471	2 times	+ 235863·066	
+ 1290	1 time	+ 1290438·27	
− 1457042)		− 415893577·	take
		− 415978976·	from
		− 85399	(+ ↓0,0,0,0,5,8,
		− 72852	
		− 12547	
		− 11656	
		− 891	

If the process be continued another step, the value of x will be found to be $20 \downarrow 1,3,2,5,5,8,7,6,$ which value might be found under many forms; for example:—

Common number.		Dual numbers.		Dual log. of x.
$x = 22\cdot7246714 =$	{	$22 \downarrow 0,3,2,5,5,8,7,5,$ $20 \downarrow 1,3,2,5,5,8,7,6,$ $15 \downarrow 4,3,4,3,1,1,3,8,$ $12 \downarrow 6,6,6,9,8,4,4,7,$ &c. &c. &c.	}	$= 312345120,$

Whence it is evident that x may be found by putting any number from 22 to 12 for r; 20 is selected, because its square, cube, &c., are easily obtained and operated with. To determine a value of x in such equations as the given ones, by any other known method, would be almost impossible, on account of the laborious calculations and other perplexing circumstances involved.

EXPONENTIAL EQUATIONS.

62. Given $x^x = 8\cdot$ to find the value of x.

Logarithms of any system being employed, it is well known that

$$x \log x = 8\,;$$

and it will be presently shown that

$$(2\cdot 38842348)^{2\cdot 38842348} = 8.$$

$$\therefore\ \downarrow,(2\cdot 38842348) \times 2\cdot 38842348 = \downarrow,(8\cdot).$$

But $\qquad 2 = \downarrow 7,2,6,0,7,8,2,6, = \overset{8}{\downarrow}\ 69314718,$

or $\qquad \downarrow,(2\cdot) = 69314718,$

since the dual digits reduced to the 8th position is termed the dual logarithm.

$$\text{Dual log of } 2, \quad \text{or} \quad \downarrow,(2\cdot) = 69314718,$$

and because $\qquad 2^3 = 8,$

therefore, $\quad 207944154, = 3\downarrow,(2\cdot) =$ the dual log of 8.

Again, $2\cdot 38842348 = 2\downarrow 1,8,2,5,7,4,5,3, = \overset{8}{\downarrow}\ \overline{87063353,}$

$$\therefore\ \downarrow,(2\cdot 38842348) = 87063353,$$

then 87063353, multiplied by $2\downarrow 1,8,2,5,7,4,5,3,$ or its equal $2\cdot 38842348$, must give $2\cdot 07944154$, if $2\cdot 38842348$ be the value of x.

Proof.

The details of the work.

8 7 0 6 3 3 5 3
2

1 7 4 1 2 6 7 0 6 .
1 7 4 1 2 6 7 1

1 9 1 5 3 9 3 7 7 . 2 ↓ 1,
1 5 3 2 3 1 5 0 .
5 3 6 3 1 0 .
0 7 2 6 .
1 3 4 .
1 .

2 0 7 4 0 9 6 9 8 2 ↓ 1,8,
4 1 4 8 1 9
2 0 7

2 0 7 8 2 4 7 2 4 2 ↓ 1,8,2,
1 0 3 9 1 2
2 1

2 0 7 9 2 8 6 5 7 2 ↓ 1,8,2,5,
1 4 5 5 5
8 3,2
1 0 4
6

↓, (8) = 2 0 7 9 4 4 1 5 4, 2 ↓ 1,8,2,5,7,4,5,3,

63. Therefore, the dual log of 2·38842348 multiplied by 2·38842348 gives the dual log of 8, and hence, 2·38842348 is the value of x in the equation $x^x = 8·$ true to nine places of figures.

It is evident that a formula, to be established presently, which in all cases reverses the above direct process, will give the value of x in the general equation $x^x = a$. Since, an independent and direct solution of this equation has defied all attempts of mathematicians by arts previously known, it may

be necessary, before delivering the general formula, to give at length the solutions of one or two particular examples, to prevent any misunderstanding.

$\downarrow, (2^{\cdot}) = 69314718,$	$\therefore\ 2\downarrow, (2^{\cdot}) = 138629436,$
$\downarrow, (3^{\cdot}) = 109861229,$	$\therefore\ 3\downarrow, (3^{\cdot}) = 329583687,$
$\downarrow, (4^{\cdot}) = 138629436,$	$\therefore\ 4\downarrow, (4^{\cdot}) = 554517744,$
$\downarrow, (5^{\cdot}) = 160943792,$	$\therefore\ 5\downarrow, (5^{\cdot}) = 804718960,$
$\downarrow, (6^{\cdot}) = 179175948,$	$\therefore\ 6\downarrow, (6^{\cdot}) = 1075055688,$
$\downarrow, (7^{\cdot}) = 194591016,$	$\therefore\ 7\downarrow, (7^{\cdot}) = 1362137112,$
$\downarrow, (8^{\cdot}) = 207944154,$	$\therefore\ 8\downarrow, (8^{\cdot}) = 1663553232,$
$\downarrow, (9^{\cdot}) = 219722458,$	$\therefore\ 9\downarrow, (9^{\cdot}) = 1977502122,$
$\downarrow, (10^{\cdot}) = 230258509,$	$\therefore\ 10\downarrow, (10^{\cdot}) = 2302585090,$
$\downarrow, (11^{\cdot}) = 239789528,$	$\therefore\ 11\downarrow, (11^{\cdot}) = 2637684808,$

These numbers may be expressed under the general form $n\downarrow, (n)$; but few of them are required. They are used in a manner similar to that in which the squares and cubes of the nine digits are employed in extracting the square and cube roots of common numbers. It will be found convenient to arrange the well-known dual numbers $\downarrow 1$, $\downarrow 0,1$, $\downarrow 0,0,1$, &c. in the following order :—

$$1{\cdot}1 = \downarrow 1, = \overset{8}{\downarrow} 9531018, \quad \text{or,} \quad \downarrow, (1{\cdot}1) = 9531018, \quad (a).$$

$$1{\cdot}01 = \overset{2}{\downarrow} 1, = \overset{8}{\downarrow} 995033, \quad \text{or,} \quad \downarrow, (1{\cdot}01) = 995033, \quad (b).$$

$$1{\cdot}001 = \overset{3}{\downarrow} 1, = \overset{8}{\downarrow} 99950, \quad \text{or,} \quad \downarrow, (1{\cdot}001) = 99950, \quad (c).$$

$$1{\cdot}0001 = \overset{4}{\downarrow} 1, = \overset{8}{\downarrow} 10000, \quad \text{or,} \quad \downarrow, (1{\cdot}0001) = 10000, \quad (d).$$

$$1{\cdot}00001 = \overset{5}{\downarrow} 1, = \overset{8}{\downarrow} 1000, \quad \text{or,} \quad \downarrow, (1{\cdot}00001) = 1000, \quad (e).$$

&c. &c. &c. &c. &c.

Given $x^x = 8$, to find the value of x.

If $x^x = \text{N}$, then $x\downarrow, (x) = \downarrow, (\text{N})$;

In the given example $x\downarrow, (x) = \downarrow, (8) = 207944154,$.

Since $2\downarrow,(2) = 138629436$, and $3\downarrow,(3) = 329583687$, it is evident that the value of x is between 2 and 3, therefore the process is commenced with $2\downarrow,(2)$, represented by $n\downarrow,(n)$.

$\frac{n\downarrow,(n)}{10}$ 13862943
190..... a_1
328.....)

$\frac{\downarrow,(N_1)}{10^2}$ 1734606
2189... b_1
3923...)

$\frac{\downarrow,(N_2)}{10^3}$ 206796
238109 c_1
444...)

$\frac{\downarrow,(N_3)}{10^4}$ 20768
23871 d_1
44639)

$\downarrow,(N)$ 2079.....
$n\downarrow,(n)$ 1386.....
693.....
328..... $(\downarrow 1, = \downarrow u_1$

$\downarrow,(N)$ 2079.....
$\downarrow,(N_1)$ 1734.....
343..... $(\downarrow^2 8, = \downarrow u_2,$

$\downarrow,(N)$ 207944...
$\downarrow,(N_2)$ 206796...
1148...
888... $(\downarrow^3 2, = \downarrow u_3,$

$\downarrow,(N)$ 207944...
$\downarrow,(N_3)$ 207687...
257...
223... $(\downarrow^4 5, = \downarrow u_4,$

$\downarrow,(2)$ 69314718
2
138629436 $n\downarrow,(n)$
19062036 u_1a_1
157691472
15769147
173460619 $\downarrow,(N_1)$
17512584 u_2b_1
190973203.
15277856.
534725.
10695.
134.
1.
206796614 $\downarrow,(N_2)$
476218 u_3c_1

$9531018 = a$
2
19062036 a_1

$995033 = b$
2
1990066
199007
2189073 b_1

$99950 = c$
2
199900
19990
219890
17591
616
12
238109 c_1

```
1 0 0 0 0 = d
        2
---------
2|0|0|0|0
 |2|0|0|0
---------
2 2|0 0|0 .
  1|7 6|0 .
     6|2 .
       |1 .
---------
2 3 8|2 3 .
     |4 8 .
---------
2 3 8 7 1    d₁

    1 0 0 0 = e
n         2
        ---------
↓ u₁,   2|0|0|0
         |2|0|0
        ---------
↓ u₂,   2 2|0 0
          1|7 6
             6
        ---------
↓ u₃,   2 3 8|2
             |5
        ---------
↓ u₄,   2 3 8 7|
              1|
        ---------
        2 3 8 8    e₁
```

```
2 0 7 2 7 2|8 3 2
     |4 1 4|5 4 6
           |2 0 7
-----------------
2 0 7 6 8 7 5 8 5   ↓, (N₃)
    1 1 9 3 5 5     u₄d₁
-----------------
2 0 7 8 0 6 9 4|0
     1|0 3 9 0|3    ↓ u₄,
             2|1
-----------------
2 0 7 9 1 0 8 6 4   ↓, (N₄)
```

```
↓,(N₄)   2079              ↓, (N)   207944154
 -----   2388        e₁    ↓, (N₄)  207910864
  10⁵    ----                       ---------
         4467 )                         33290
         . . . ↑                        31269 (↓⁵ 7,4,5,3,
                                        -----
                                         2021
                                         1786
                                         ----
                                          235
                                          223
                                          ---
                                           12
                                           12
                                           --
```

Each step is obtained from what precedes it, thus e_1 is found from knowing e and $n \downarrow u_1, u_2, u_3, u_4$; $\downarrow, (N_4)$ is readily obtained since u_4 and d_1 have been previously determined. A similar treatment is applied at each step in finding the digits $u_1, u_2, u_3, \ldots$ The digits $\overset{5}{\downarrow}\, 7,4,5,3,$ are found by common division.

$$\therefore\quad x = 2 \downarrow 1,8,2,5,7,4,5,3 = 2{\cdot}38842348.$$

P

Ex. 6. *The population of the earth at present* (1866) *is estimated at* 1,123,477,000 *souls; find the value of* x *in the equation*

$$x^x = 1123477000.$$

$$x^x = 1123477000 = 10^9 \downarrow 1,2,1,2,1,8,0,1, = \downarrow 2083969416,$$

$$\therefore \quad x\downarrow,(x) = \downarrow,(1123477000) = 2083969416, = \downarrow,(\mathrm{N})$$

Because $9\downarrow,(9) = 1977502122$, and $10\downarrow,(10) = 2302585090$, x must be greater than 9, but less than 10.

$$\begin{array}{lrl}
\frac{n\downarrow,(n)}{10} & 197750212 & \\
 & 85779162 & a_1 \\
\hline
 & 282\ldots\ldots) &
\end{array}
\qquad
\begin{array}{lrl}
\downarrow,(\mathrm{N}) & 2083\ldots\ldots & \\
n\downarrow,(n) & 1977\ldots\ldots & \\
\hline
 & 106\ldots\ldots & (\downarrow 0, = \downarrow u_1
\end{array}$$

$$\begin{array}{lrl}
\frac{n\downarrow,(n)}{10^2} & 19775021 & \\
 & 8955\ldots & b_1 \\
\hline
 & 2873\ldots) &
\end{array}
\qquad
\begin{array}{lrl}
\downarrow,(\mathrm{N}) & 20839\ldots\ldots & \\
n\downarrow,(n) & 1977\ldots\ldots & \\
\hline
 & 106\ldots\ldots & (\downarrow^2 3, = \downarrow u_2
\end{array}$$

$$\begin{array}{lrl}
\frac{\downarrow,(\mathrm{N}_2)}{10^3} & 2065102 & \\
 & 926808 & c_1 \\
\hline
 & 299\ldots) &
\end{array}
\qquad
\begin{array}{lrl}
\downarrow,(\mathrm{N}) & 208396\ldots & \\
\downarrow,(\mathrm{N}_2) & 206510\ldots & \\
\hline
 & 1886\ldots & (\downarrow^3 6, = \downarrow u_3
\end{array}$$

$$9\downarrow,(9) = 1977502122,\ n\downarrow,(n)$$

$$\begin{array}{rl}
1977502122 & n\downarrow,(n) \\
26865891 & u_2 b_1 \\
\hline
20043680\ 13 & \\
60\ 13\ 10\ 40 & \\
60\ 13\ 10 & \\
20\ 04 & \\
\hline
2065102367 & \downarrow,(\mathrm{N}_2) \\
5560848 & u_3 c_1 \\
\hline
\end{array}$$

$$\begin{array}{rl}
9531018 & = a \\
9 & \\
\hline
85779172 & a_1
\end{array}
\qquad
\begin{array}{rl}
995033 & = b \\
9 & \\
\hline
8955297 & b_1
\end{array}$$

$\dfrac{\downarrow,(N_3)}{10^4}$

208311		$\downarrow,(N)$	2083969 . . .
93285	d_1	$\downarrow,(N_3)$	2083118 . . .
301 . . .)			851 . . . $(\downarrow^4 2, = \downarrow u_4$

$\dfrac{\downarrow,(N_4)}{10^5}$

20837		$\downarrow,(N)$	2083969 . . .
9331	e_1	$\downarrow,(N_4)$	2083721 . . .
30168)			248 . . . $(\downarrow^5 8, = \downarrow u_5$

$\dfrac{\downarrow,(N_5)}{10^6}$

2084		$\downarrow,(N)$	2083969416
933	f_1	$\downarrow,(N_5)$	2083962905
3017)			6511 $(\downarrow^6 2,1,6,$
			6034
			477
			302
			175

20 7 | 0 6 6 | 32 1 | 5	
1 | 2 4 2 | 39 7 | 9	
3 | 10 6 | 0	
4 | 1	
20 8 3 1 1 82 9 5	$\downarrow,(N_3)$
1 8 65 7 0	$u_4 d_1$
20 8 3 | 3 0 48 | 6 5	
4 1 66 | 6 1	
2 1	
20 8 3 7 2 15 4 7	$\downarrow,(N_4)$
7 46 4 8	$u_5 e_1$
20 8 3 7 | 9 61 9 5	
1 | 6 67 0 4	
6	
20 8 3 9 6 29 0 5	$\downarrow,(N_5)$

9 99 50	$= c$
9	
89 | 95 | 50	
2 | 69 | 87	
2 | 70	
1	
92 68 08	c_1

$(10000)\times 9\downarrow 0,3,6, = 93285\, d_1$

$(1000)\times 9\downarrow 0,3,6,2, = 9331\, e_1$

$(100)\times 9\downarrow 0,3,6,2,8, = 933\, f_1$

$$\therefore \quad x = 9\downarrow 0,3,6,2,8,2,1,6, = 9{\cdot}33111684.$$

64. General solution of the equation

$$x^x = \mathrm{N}.$$

If $\quad x = n \downarrow u_1, u_2, u_3, \ldots = n\{1 \dotplus [u_1 \dotplus [u_2 \dotplus [u_3 \dotplus \ldots\}$

Then
$$\begin{aligned}
n \downarrow u_1, u_2, &= n\{1 \dotplus [u_1 \dotplus [u_2\} \\
&= n\{1 \dotplus [u_1\}\ \{1 \dotplus [u_2\} \\
&= n\{1 \dotplus [u_2\}\ \{1 \dotplus [u_1\} \\
n \downarrow u_1, u_2, u_3, &= n\{1 \dotplus [u_1 \dotplus [u_2 \dotplus [u_3\} \\
&= n\{1 \dotplus [u_1 \dotplus [u_2\}\ \{1 \dotplus [u_3\} \\
&= n\{1 \dotplus [u_1\}\ \{1 \dotplus [u_2\}\ \{1 \dotplus [u_3\} \\
&= n\{1 \dotplus [u_1 \dotplus [u_3\}\ \{1 \dotplus [u_2\} \\
&\&c. \qquad \&c. \\
n \downarrow u_1, u_2, u_3, u_4, &= n\{1 \dotplus [u_1 \dotplus [u_2 \dotplus [u_3 \dotplus [u_4\} \\
&= n\{1 \dotplus [u_1 \dotplus [u_2 \dotplus [u_3\}\ \{1 \dotplus [u_4\} \\
&\&c. \qquad \&c.
\end{aligned}$$

Multiply

$$x = n \downarrow u_1, u_2, u_3, \ldots\ldots n\{1 \dotplus [u_1 \dotplus [u_2 \dotplus [u_3 \dotplus [u_4 \dotplus \ldots\ldots\}$$

by $\qquad \log x = \downarrow,(n) + u_1 a + u_2 b + u_3 c + \ldots$

$$\begin{aligned}
&n \downarrow,(n)\{1 \dotplus [u_1 \dotplus [u_2 \dotplus [u_3 \dotplus \ldots\ldots\} \\
&\quad + n u_1 a\{1 \dotplus [u_1 \dotplus [u_2 \dotplus [u_3 \dotplus \ldots\ldots\} \\
&\qquad + n u_2 b\{1 \dotplus [u_1 \dotplus [u_2 \dotplus [u_3 \dotplus \ldots\ldots\} \\
&\qquad\quad + n u_3 c\{1 \dotplus [u_1 \dotplus [u_2 \dotplus [u_3 \dotplus \ldots\ldots\} \\
&\qquad\qquad + \quad \&c. \qquad \&c.
\end{aligned}$$

Multiply

$$n\{1 \underset{\downarrow}{+} [u_1\}$$

by $$\downarrow,(n) + u_1 a$$

$$(n\downarrow,(n) + nu_1 a)\{1 \underset{\downarrow}{+} [u_1\} = \downarrow,(\mathrm{N}_1);$$

$$na \text{ being put } = a_1$$

Multiply

$$\left\{\begin{matrix}\downarrow,(n) + u_1 a + u_2 b \\ n\{1 \underset{\downarrow}{+} [u_1 \underset{\downarrow}{+} [u_2\}\end{matrix}\right\} = \begin{matrix}\downarrow,(n) + u_1 a + u_2 b \\ n\{1 \underset{\downarrow}{+} [u_1\}\{1 \underset{\downarrow}{+} [u_2\}\end{matrix}$$

$$(\downarrow,(\mathrm{N}_1) + nu_2 b\{1 \underset{\downarrow}{+} [u_1\})\{1 \underset{\downarrow}{+} [u_2\} \text{ put} = \downarrow,(\mathrm{N}_2);$$

$$nb\{1 \underset{\downarrow}{+} [u_1\} \text{ being put } = b_1$$

Multiply

$$\downarrow,(n) + u_1 a + u_2 b + u_3 c$$

by $$n\{1 \underset{\downarrow}{+} [u_1 \underset{\downarrow}{+} [u_2 \underset{\downarrow}{+} [u_3\} = n\{1 \underset{\downarrow}{+} [u_1 \underset{\downarrow}{+} [u_2\}\{1 \underset{\downarrow}{+} [u_3\}$$

$$(\downarrow,(\mathrm{N}_2) + nu_3 c\{1 \underset{\downarrow}{+} [u_1 \underset{\downarrow}{+} [u_2 \underset{\downarrow}{+} [u_3\})\{1 \underset{\downarrow}{+} [u_3\} = \downarrow,(\mathrm{N}_3);$$

$$nc\{1 \underset{\downarrow}{+} [u_1 \underset{\downarrow}{+} [u_2 \underset{\downarrow}{+} [u_3\} \text{ being put } = c_1$$

Multiply

$$\downarrow,(n) + u_1 a + u_2 b + u_3 c + u_4 d$$

by $$n\{1 \underset{\downarrow}{+} [u_1 \underset{\downarrow}{+} [u_2 \underset{\downarrow}{+} [u_3 \underset{\downarrow}{+} [u_4\} = n\{1 \underset{\downarrow}{+} [u_1 \underset{\downarrow}{+} [u_2 \underset{\downarrow}{+} [u_3\}\{1 \underset{\downarrow}{+} [u_4\}$$

$$(\downarrow,(\mathrm{N}_3) + nu_4 d\{1 \underset{\downarrow}{+} [u_1 \underset{\downarrow}{+} [u_2 \underset{\downarrow}{+} [u_3\})\{1 \underset{\downarrow}{+} [u_4\} = \downarrow,(\mathrm{N}_4);$$

$$nd\{1 \underset{\downarrow}{+} [u_1 \underset{\downarrow}{+} [u_2 \underset{\downarrow}{+} [u_3\} \text{ being put } = d_1$$

In the same manner the development may be continued.

When $\uparrow,(\mathrm{N})$; $n\downarrow,(n)$ and a_1 become known, then u_1 may be found.

If $x = n \downarrow u_1$ then $n\{1 \ddagger [u_1\}$ multiplied by $\downarrow,(n) + u_1 a = \downarrow,(\text{N})$.

$$\therefore \quad n \downarrow,(n)\{1 \ddagger [u_1\} + nu_1 a\{1 \ddagger [u_1\} = \downarrow,(\text{N})$$

$$\therefore \quad n \downarrow,(n) \ddagger n \downarrow,(n)[u_1 + nu_1 a \ddagger \ldots = \downarrow,(\text{N})$$

$$\therefore \quad \ddagger n \downarrow,(n)[u_1 + nau_1 \ddagger \ldots = \downarrow,(\text{N}) - n \downarrow,(n)$$

and $\therefore$ u_1 may be determined from

$$\frac{\downarrow,(\text{N}) - n \downarrow,(n)}{+ \frac{n \downarrow,(n)}{10} + na} = \frac{\downarrow,(\text{N}) - n \downarrow,(n)}{+ \frac{n \downarrow,(n)}{10} + a_1}$$

Then $\downarrow,(\text{N})$; $\downarrow,(\text{N}_1)$; and b_1 become known, and u_2 may be found, for if $x = n \downarrow u_1, u_2$, then $\downarrow(n) + u_1 a + u_2 b$ multiplied by

$$n\{1 \ddagger [u_1 + [u_2\} = (\downarrow,(\text{N}_1) + u_2 b_1)\{1 \ddagger [u_2\}$$

$$= \downarrow,(\text{N}_1)\{1 \ddagger [u_2\} + u_2 b_1\{1 \ddagger [u_2\}$$

$$= \downarrow,(\text{N}_1) \ddagger \downarrow,(\text{N}_1)[u_2 + u_2 b_1 \ddagger \ldots = \downarrow,(\text{N})$$

$$\therefore \quad \ddagger \downarrow,(\text{N}_1)[u_2 + u_2 b_1 \ddagger \ldots = \downarrow,(\text{N}) - \downarrow(\text{N}_1)$$

and $\therefore$ u_2 becomes known from

$$\frac{\downarrow,(\text{N}) - \downarrow,(\text{N}_1)}{+ \frac{\downarrow,(\text{N})}{10^2} + b_1}$$

Then $\downarrow,(\text{N})$; $\downarrow,(\text{N}_2)$; and c_1 becomes known, and u_3 may be found, for if $x = n \downarrow u_1, u_2, u_3$, then $\downarrow,(n) + u_1 a + u_2 b + u_3 c$ multiplied by $\{n1 \ddagger [u_1 \ddagger [u_2 \ddagger [u_3\}$

$$= (\downarrow,(\text{N}_2) + u_3 c_1)\{1 \ddagger [u_3\}$$

$$= \downarrow,(\text{N}_2)\{1 \ddagger [u_3\} + u_3 c_1\{1 \ddagger [u_3\} = \downarrow,(\text{N})$$

$$= \downarrow,(\text{N}_2) \ddagger \downarrow,(\text{N}_2)[u_3 + u_3 c_1 \ddagger \ldots = \downarrow,(\text{N})$$

$$\therefore \quad \ddagger \downarrow,(\text{N}_2)[u_3 + c_1 u_3 \ddagger \ldots = \downarrow,(\text{N}) - \downarrow,(\text{N}_2)$$

and $\therefore$ u_3 becomes known from

$$\frac{\downarrow,(\text{N}) - \downarrow,(\text{N}_2)}{+ \frac{\downarrow,(\text{N}_2)}{10^3} + c_1}$$

In the same manner u_4, u_5, &c. may be found, and

$$x = n \downarrow u_1, u_2, u_3, \ldots$$

becomes known.

Although the value of

$$\overset{+}{\downarrow} n \downarrow, (n)\,[u_1 + u_1 a_1, \overset{+}{\downarrow} \ldots = \downarrow, (\mathrm{N}) - n \downarrow, (n)$$

may be accurately found, yet it may be put under the form

$$\left(\frac{n \downarrow, (n)}{10} + a_1\right) u_1 = \downarrow, (\mathrm{N}) - n \downarrow, (n)$$

for if u_1 be taken too great or too small the process may still be continued, as the excess or defect will be corrected by the succeeding steps. The same remark applies to

$$\overset{+}{\downarrow} \downarrow, (\mathrm{N}_1)\,[u_2 + p_1 u_2 \overset{+}{\downarrow} \ldots = \downarrow, (\mathrm{N}) - \downarrow, (\mathrm{N}_1)$$

which is put under the form

$$(+ \frac{\downarrow, (\mathrm{N})}{10^2} + b_1) u_2 = \downarrow, (\mathrm{N}) - \downarrow, (\mathrm{N}_1)$$

to determine u_2; and so on with respect to u_3, u_4, &c.

Consequently the process is continuous without interruption, since x may be represented under a vast number of dual forms, all amounting to the same natural number. For example, x was found under the form $2 \downarrow 1,8,2,5,7,4,5,3$, in the equation $x^x = 8$; but x might be found under the form '0'1'3'0'8'2'1'7 $1 \uparrow 2$, each of these dual numbers when reduced becomes 2·33842348.

CHAPTER IV.

SPECIAL TREATMENT OF THE DESCENDING BRANCH OF DUAL ARITHMETIC.

65. ALTHOUGH we have defined and described the nature of this branch of dual arithmetic, yet, to avoid complication, the special treatment and practical application of this branch of the art have been postponed until now; the method thus pursued will be found to possess many advantages.

When both branches are judiciously combined, great power is gained and much labour saved. Operations, that might be cumbersome with either branch, may be rendered simple and concise by combining both.

In the descending branch, the bases $\cdot9$; $\cdot99$; $\cdot999$; &c. or $1 - \cdot1$; $1 - \cdot01$; $1 - \cdot001$; &c. are employed in a manner similar to that in which the bases $1\cdot1$; $1\cdot01$; $1\cdot001$; &c. or $1 + \cdot1$; $1 + \cdot01$; $1 + \cdot001$; &c. are engaged in the ascending branch. See Articles (1) to (7), pp. 1 to 6.

$$(1 - \cdot1)^6 (1 - \cdot01)^4 (1 - \cdot001)^3 (1 - \cdot0001)^5$$
$$= (\cdot9)^6 (\cdot99)^4 (\cdot999)^3 (\cdot9999)^5 = \cdot508715701$$

and is represented thus,

$$,6,4,3,5,0,0,0,0 \dagger = \cdot508715701$$

66. DETAILS OF TWO METHODS OF REDUCTION.

By continually Subtracting.

	1·000000000
	100000000
	900000000
	90000000
	810000000
	8100000
’6 $\uparrow_1$	729000000
	72900000
	656100000
	65610000
	590490000
	59049000
	53|1441000
	5314410
’4 $\uparrow_2$	526126590
	5261266
	520865324
	5208653
	515656671
	5156567
	510|500104
	510500
’3 $\uparrow$	509989604
	509989
	509479615
	509479
	5089|70136

By Binomial Coefficients, or Operative Numbers.

	1000000000	+
	600000000	−
	150000000	+
’6 $\uparrow_1$	20000000	−
	1500000	+
	60000	−
	1000	+
	531441000 .	+
	212576400 .	−
’4 $\uparrow_2$	318865 .	+
	2126 .	−
	5 .	+
	510500104	+
’3 $\uparrow_3$	1531500	−
	1532	+
	1	−
	5089701355	+
’5 $\uparrow_4$	254485	−
	51	+
	·508715701	

In particular cases, the reduction will be found more convenient by operating with binomial coefficients, when proper contractions are employed; but the reduction by continually subtracting is best suited for forming a table, or to obtain a consecutive series of these numbers. The following table (A), will be

5089 | 70136
50897

'5$_4$ ↑ 5089 19239
50892

5088 68347
50887

5088 17460
50882

5087 66578
50877

·5087 15701

found useful in reducing numbers to a dual form; it is constructed with the greatest ease, each natural or common number being obtained from the preceding one, by subtracting each digit of the line above with 0, suppose after it, from the digit to its left. For example, take 6561, which imagine to be 65610.

Then 1 from 0 gives 9, carry 1 to
6 = 7 from 1 gives 4, carry 1 to
5 = 6 from 6 gives 0, carry 0 to
6 = 6 from 5 gives 9, and 1 from 6
gives 5; thus 59049 is instantly found.

TABLE (A).

	Common number.	Dual number.	Dual log negative.
$(\cdot 9)^1$	·9	'1 . . . ↑	'105360 52 ↑
$(\cdot 9)^2$	·81	'2 . . . ↑	'21072104 ↑
$(\cdot 9)^3$	·729	'3 . . . ↑	'31608156 ↑
$(\cdot 9)^4$	·6561	'4 . . . ↑	'42144208 ↑
$(\cdot 9)^5$	·59049	'5 . . . ↑	'52680260 ↑
$(\cdot 9)^6$	·531441	'6 . . . ↑	'63216312 ↑
$(\cdot 9)^7$	·4782969	'7 . . . ↑	'73752364 ↑
$(\cdot 9)^8$	·43046721	'8 . . . ↑	'84288416 ↑
$(\cdot 9)^9$	·387420489	'9 . . . ↑	'94824468 ↑

67. The dual logarithms are found by multiplying 10536052 by 1, 2, 3, &c. respectively.

$$'1'0'0'0'0'0'0'0\uparrow = '10536052\underset{8}{\uparrow} = '1\underset{1}{\uparrow}$$

$$'0'1'0'0'0'0'0'0\uparrow = '1005034\underset{8}{\uparrow} = '1\underset{2}{\uparrow}$$

$$'0'0'1'0'0'0'0'0\uparrow = '100050\underset{8}{\uparrow} = '1\underset{3}{\uparrow}$$

$$'0'0'0'0'1'0'0'0'0\uparrow = '10000\underset{8}{\uparrow} = '1\underset{4}{\uparrow}$$

&c. &c. &c.

Because $(\cdot 9)\downarrow 1,1,0,1,0,0,0,1, = 1\cdot$

and by (12), $\downarrow 1,1,0,1,0,0,0,1, = \downarrow 10536052,$

and $\therefore\ '10536052\underset{8}{\uparrow} = \overset{8}{\downarrow} - 10536052,$

or $'10536052\uparrow = \downarrow - 10536052,$

Dual log of $(\cdot 9)$, or $\downarrow, (\cdot 9) = -\ 10536052,$

A dual number reduced to the eight position being termed a a dual logarithm, the 8, marking this position, for the sake of brevity, is omitted in the descending as well as in the ascending branch.

$$-\ 10536052, = (\cdot 9)'\uparrow = '10536052$$

Again,

$(\cdot 99)\downarrow 0,1,0,1,0,0,0,1, = 1\cdot$ but $\downarrow 0,1,0,1,0,0,0,1, = \downarrow 1005034,$

$$\therefore\ '1005034\uparrow = \downarrow - 1005034,$$

or $'1\underset{2}{\uparrow} = '1005034\underset{8}{\uparrow}$

$\downarrow, (\cdot 99) = -\ 1005034,$ written $'1005034$

And again, because

$(\cdot 999)\downarrow 0,0,1,0,0,1,0,0, = 1\cdot$ but $\downarrow 0,0,1,0,0,1,0,0, = \downarrow 100050,$

$$\therefore\ '100050\uparrow = \downarrow - 100050,$$

or $'1\underset{3}{\uparrow} = '100050\underset{8}{\uparrow}$

&c. = &c.

Then it is readily shown that

$$\cdot 508715701 = '67586598\uparrow = '6'4'3'5'0'0'0'0\uparrow$$

'6'4'3'5'0'0'0'0↑			
3 2 1 6 3 1 2	six times	536052
2 0 1 3 6	four times	5034
1 5 0	three times	50
'6 7 5 8 6 5 9 8↑			

So that any descending dual number may be reduced to its eight position by adding to the dual number, considered as a common number, 536052 multiplied by the first dual digit, 5034 multiplied by the second, and 50 by the third dual digit. See Rule, Article (14), page 19.

Ex. Let it be required to reduce '6'6'0'6'8'2'0'3↑ to its eight position, or to its dual logarithm.

66068203		
3216312	six times	536052
30204	six times	5034
69314719		

$$\therefore \quad '69314719\underset{8}{\uparrow} = '6'6'0'6'8'2'0'3\uparrow$$

This Rule is easily reversed, and a descending dual logarithm reduced to a descending dual number. See Rule, Article (15), page 21.

Examples.

1. Reduce the descending dual logarithm '69314718↑ to its corresponding dual number.

$$
\begin{array}{rl}
'69314719 & \\
3216312 & \text{six times } 536052 \\
\hline
66098407 & \\
30204 & \text{six times } \quad 5034 \\
\hline
'6'6'0'6'8'2'0'3\uparrow &
\end{array}
$$

$$\therefore \quad '69314719\uparrow = '6'6'0'8'2'0'3\uparrow$$

As the third dual digit is 0, no multiple of 50 has to be subtracted.

2. Reduce the descending dual logarithm '67586598↑ to its corresponding dual number.

$$
\begin{array}{rl}
'67586598 & \\
3216312 & \text{six times } 536052 \\
\hline
'64370286 & \\
20136 & \text{four times } \quad 5034 \\
\hline
'64350150 & \\
150 & \text{three times } \quad 50 \\
\hline
'6'4'3'5'0'0'0'0\uparrow &
\end{array}
$$

$$\therefore \quad '67586598\uparrow = '6'4'3'5'0'0'0'0\uparrow$$

68. If dual logarithms of the ascending branch be considered positive, those of the descending branch must be considered negative, and *vice versâ.*

Because $\dfrac{1}{(1\cdot 00000001)} = (\cdot 99999999)$ very nearly.

$$\therefore \quad \frac{1}{(1\cdot 00000001)^p} = (\cdot 99999999) \;\; ;$$

But

$$\frac{1}{(1\cdot00000001)^p} = \overset{8}{\downarrow} -p, ; \quad \text{and} \quad (\cdot99999999)^p = {}'p\underset{8}{\uparrow} ;$$

$$\therefore \quad {}'p\underset{8}{\uparrow} = \overset{8}{\downarrow} -p,$$

Actual division of 1· *by* 1·00000001 ;

```
1·00000001 ) 1·000000000 ( ·99999999
              900000009
              ---------
              999999910
              900000009
              ---------
               999999010
               900000009
               ---------
                999990010
                900000009
                ---------
                 999900010
                 900000009
                 ---------
                  999000010
                  900000009
                  ---------
                   990000010
                   900000009
                   ---------
                    900000010
                    900000009
                    ---------
                    Remainder 1
```

Hence, the log of ·508715701, found by the descending process, is the same as the log of ·508715701, with a negative sign, found by the ascending process; for

$$2(\cdot508715701) = 1\cdot017431402 = \downarrow 0,1,7,3,3,4,4,0,$$

and $$\downarrow 0,1,7,3,3,4,4,0, = \downarrow 1728123, ;$$

But

$$\downarrow,(2) + \downarrow,(\cdot 508715701) = \downarrow,(1\cdot 017431402)$$

$$\therefore \quad \downarrow,(\cdot 508715701) = \downarrow,(1\cdot 017431402) - \downarrow,(2)$$

$$\begin{aligned} -\downarrow,(2) \quad &= -\ 69314718, \\ \downarrow,(1\cdot 0174\ldots) &= \quad 1728123, \\ \hline \therefore \quad \cdot 508715701 &= \downarrow -\ 67586595, \end{aligned}$$

It was before shown that

$$\cdot 508715701 = {}'67586598\uparrow$$

results that may be said to coincide.

Table (B) is very easily constructed by subtracting as follows; and adding the constant log '1005034 ↑.

$$\begin{array}{lcl} \cdot 99000000 = {}'0'1\ldots\uparrow & = & {}'1005034\uparrow \\ \underline{\quad 990000} & & \\ \cdot 98010000 = {}'0'2\ldots\uparrow & = & {}'2010068\uparrow \\ \underline{\quad 980100} & & \\ \cdot 97029900 = {}'0'3\ldots\uparrow & = & {}'3015102\uparrow \\ \underline{\quad 970299} & & \\ \cdot 96059601 = {}'0'4\ldots\uparrow & = & {}'4020136\uparrow \\ \underline{\quad 960596} & & \\ \cdot 95099005 = {}'0'5\ldots\uparrow & = & {}'5025170\uparrow \\ \&c. \qquad \&c. & & \&c. \end{array}$$

The numbers corresponding to '1'0 . . ↑ '2'0 . . ↑ '3'0 . . ↑ &c. are obtained from Table (A).

Table (B).

Common Numbers.	Descending Dual Numbers.	Dual Logarithms; negative.
·99000000	'0'1 . . ↑	'1005034
·98010000	'0'2 . . ↑	'2010068
·97029900	'0'3 . . ↑	'3015102
·96059601	'0'4 . . ↑	'4020136
·95099005	'0'5 . . ↑	'5025170
·94148015	'0'6 . . ↑	'6030204
·93206535	'0'7 . . ↑	'7035238
·92274470	'0'8 . . ↑	'8040272
·91351725	'0'9 . . ↑	'9045306
·90000000	'1'0 . . ↑	'10536052
·89100000	'1'1 . . ↑	'11541086
·88209000	'1'2 . . ↑	'12546120
·87326910	'1'3 . . ↑	'13551154
·86453641	'1'4 . . ↑	'14556188
·85589105	'1'5 . . ↑	'15561222
·84733214	'1'6 . . ↑	'16566256
·83885882	'1'7 . . ↑	'17571290
·83047023	'1'8 . . ↑	'18576324
·82216553	'1'9 . . ↑	'19581358
·81000000	'2'0 . . ↑	'21072104
·80190000	'2'1 . . ↑	'22077138
·79388100	'2'2 . . ↑	'23082172
·78594219	'2'3 . . ↑	'24087206
·77808277	'2'4 . . ↑	'25092240
·77030194	'2'5 . . ↑	'26097274
·76259892	'2'6 . . ↑	'27102308
·75497293	'2'7 . . ↑	'28107342
·74742320	'2'8 . . ↑	'29112376

TABLE (B)—*continued.*

Common Numbers.	Descending Dual Number.	Dual Logarithms; negative.
·73994897	'2'9 . . ↑	'30117410
·72900000	'3'0 . . ↑	'31608156
·72171000	'3'1 . . ↑	'32613190
·71449290	'3'2 . . ↑	'33618224
·70734797	'3'3 . . ↑	'34623258
·70027449	'3'4 . . ↑	'35628292
·69327175	'3'5 . . ↑	'36633326
·68633903	'3'6 . . ↑	'37638360
·67947564	'3'7 . . ↑	'38643394
·67268088	'3'8 . . ↑	'39648428
·66595407	'3'9 . . ↑	'40653462
·65610000	'4'0 . . ↑	'42144208
·64953900	'4'1 . . ↑	'43149242
·64304361	'4'2 . . ↑	'44154276
·63661317	'4'3 . . ↑	'45159310
·63024704	'4'4 . . ↑	'46164344
·62394457	'4'5 . . ↑	'47179378
·61770512	'4'6 . . ↑	'48184412
·61152807	'4'7 . . ↑	'49189446
·60541279	'4'8 . . ↑	'50194480
·59935866	'4'9 . . ↑	'51199514
·59049000	'5'0 . . ↑	'52680260
·58458510	'5'1 . . ↑	'53685294
·57873925	'5'2 . . ↑	'54690328
·57295186	'5'3 . . ↑	'55695362
·56722234	'5'4 . . ↑	'56700396
·56155012	'5'5 . . ↑	'57705430
·55593462	'5'6 . . ↑	'58710464
·55037527	'5'7 . . ↑	'59715498
·54487152	'5'8 . . ↑	'60720532
·53942280	'5'9 . . ↑	'61725566

TABLE (B)—*continued.*

Common Numbers.	Descending Dual Number.	Dual Logarithms; negative.
·53144100	'6'0 . . ↑	'63216312
·52612659	'6'1 . . ↑	'64221346
·52086532	'6'2 . . ↑	'65226380
·51565667	'6'3 . . ↑	'66231414
·51050010	'6'4 . . ↑	'67236448
·50539510	'6'5 . . ↑	'68241478
·50034115	'6'6 . . ↑	'69246512
·49533774	'6'7 . . ↑	'70251546
·49038436	'6'8 . . ↑	'71256580
·48548052	'6'9 . . ↑	'72261614
·47829690	'7'0 . . ↑	'73752364
·47351393	'7'1 . . ↑	'74757398
·46877879	'7'2 . . ↑	'75762432
.46409100	'7'3 . . ↑	'76767466
·45945009	'7'4 . . ↑	'77772500
·45485459	'7'5 . . ↑	'78777534
·45030604	'7'6 . . ↑	'79782568
·44580298	'7'7 . . ↑	'80787602
·44134495	'7'8 . . ↑	'81792636
·43693150	'7'9 . . ↑	'82797670
·43046721	'8'0 . . ↑	'84288416
·42616254	'8'1 . . ↑	'85293450
·42190091	'8'2 . . ↑	'86298484
·41768190	'8'3 . . ↑	'87303518
·41350508	'8'4 . . ↑	'88308552
·40937003	'8'5 . . ↑	'89313586
·40527633	'8'6 . . ↑	'90318620
·40122357	'8'7 . . ↑	'91323654
·39721133	'8'8 . . ↑	'92328088
·39323922	'8'9 . . ↑	'93333722
·38742049	'9'0 . . ↑	'94824468

Table (B) is easily extended in a similar manner by continually subtracting and adding 100050 to the logarithms as follows:—

·99900000 = 'o'o'1 . . ↑ = '100050 ↑
99900

·99800100 = 'o'o'2 . . ↑ = '200100 ↑
99800

·99700300 = 'o'o'3 . . ↑ = '300150 ↑
99700

·99600600 = 'o'o'4 . . ↑ = '400200 ↑
99601

·99501099 = 'o'o'5 . . ↑ = '500250 ↑
99501

·99401598 = 'o'o'6 . . ↑ = '600300 ↑
99402

·99302196 = 'o'o'7 . . ↑ = '700350 ↑
99302

·99202894 = 'o'o'8 . . ↑ = '800400 ↑
99203

·99103691 = 'o'o'9 . . ↑ = '900450 ↑
·99000000 = 'o'1'o . . ↑ = '1005034 ↑

&c. &c. &c.

The numbers corresponding to 'o'1 . . ↑ 'o'2 . . ↑ 'o'3 . . ↑ &c. are obtained from Table (B), and those corresponding to '1 . . ↑ '2 . . ↑ '3 . . ↑ &c. are given in table (A).

Hence, a table of descending dual numbers may be formed with great ease, in a short time, to any required extent.

OPERATIONS WITH THE DESCENDING BRANCH OF DUAL ARITHMETIC, INDEPENDENT OF THE ASCENDING BRANCH.

When operating with the descending branch alone, the digits, for the sake of convenience, may be placed to the right of the sign $\uparrow$; a dual number so expressed is sufficiently distinguished from one of the ascending branch whose digits are to the right of $\downarrow$; the point of the arrow pointing up in the former case, and down in the latter.

Examples.

1. Find the dual logarithms of $2\cdot$ by the descending process.

$$
\begin{array}{rll}
\downarrow\!\cdot 50000000 = \tfrac{1}{2} = \uparrow\,'6'6'0'6'8'2'0'3 & & \\
3\;2\;1\;6\;3\;1\;2 & \text{six times} & 536052 \\
3\;0\;2\;0\;4 & \text{six times} & 5030 \\
\hline
\uparrow_{8}\,'6\;9\;3\;1\;4\;7\;1\;9 & &
\end{array}
$$

$\therefore$ Dual log $(\tfrac{1}{2})$, written $\downarrow, (\tfrac{1}{2}) = -\,69314719, = '69314719$

$\therefore$ Dual log of $(2\cdot)$ written $\downarrow, (2) = +\,69314719$, which agrees, to a unit in the eight place, with the logarithm of the same number found by the direct process.

$\cdot 50034115 = \uparrow 6{,}6$, may be taken from Table (B) or directly calculated.

$$
\begin{array}{lr}
'6'6\ldots\uparrow = \cdot 5003|4\;1\;15\;+ \\
3|0\;0\;20\;- \\
'6\uparrow_{4} \qquad\qquad 7\;+ \\
\hline
'8\uparrow_{5} \qquad 5000\;4|1\;02 \\
4|0\;09 \\
\hline
'2\uparrow_{6} \qquad 1|02 \\
1|00 \\
\hline
'3\uparrow_{8} \qquad 2| \\
2| \\
\hline
\end{array}
$$

$\because$ $\cdot'6'6'0'6'8'2'0'3\uparrow$ above written $\uparrow'6'6'0'6'8'2'0'3 = \tfrac{1}{2}$.

2. Find the dual logarithm of $10^{\cdot}$ by the descending process.

$$2^{10} = 1024$$

$$1024 \uparrow '0'2'3'6'1'4'3'2 = 1000 = 10^3$$

$$\begin{array}{rl}
\uparrow '0'2'3'6'1'4'3'2 & \\
10068 & \text{two times } 5034 \\
150 & \text{three times } 50 \\
\hline
'2371650\uparrow &
\end{array}$$

$$\begin{array}{r}
\downarrow,(2)^{10} = 693147180, \\
'2371650 \\
\hline
3\,)\,690775530 = \downarrow,(10^3) \\
\hline
\therefore\quad \downarrow,(10) = 230258510
\end{array}$$

This result agrees with that found by the direct process to a unit in the ninth place, the logarithm before found being 230258509.

The reduction in detail.

$$\begin{array}{lrl}
'2\uparrow_{2} & 10\,|\,24\,|\,00\,|\,00\,|\,0\,.\,+ & = (2)^{10} \\
 & |\,20\,|\,48\,|\,00\,|\,0\,.\,- & \\
 & |\,10\,|\,24\,|\,0\,.\,+ & \\
\hline
'3\uparrow_{3} & 100\,|\,362\,|\,240\ \ + & \\
 & |\,301\,|\,087\ \ - & \\
 & |\,301\ \ + & \\
\hline
'6\uparrow_{4} & 1000\,|\,6145\,|\,4\ \ + & \\
 & |\,6003\,|\,7\ \ - & \\
 & 1\,|\,5\ \ + & \\
\hline
 & 10000\,|\,1432 &
\end{array}$$

$\therefore$ $'0'2'3'6'1'4'3'2 \uparrow 1024$ above written $1024 \uparrow 0,2,3,6,1,4,3,2,$ is equal $1000 = 10^3$.

3. **Find the dual logarithm of 947·01510 by the descending process.**

$$947{\cdot}01510 = (10^3)(\cdot 94701510)$$

$$\cdot 94701510 = {}'0'5'4'1'8'6'5'6\uparrow$$

25170	five times 5034
200	four times 50
'5444026↑	

↓, (10³) =	+ 690775527,
↓, (·94701510) =	'5444026
	685331501, = ↓, (947·0151)

Reducing Process.

	·947 0 1 5 1 0	given number.
'0'5 . . ↑ =	950 9 9 0 0 5 +	from table (B).
	3 8 0 3 9 6 −	
$'4\uparrow_3$	5 7 1 +	
$'1\uparrow_4$	9471 9 1 8 0	
	9 4 7 2	
	947 0 9 7 0 8	
$'8\uparrow_5$	8 1 9 8	diff,
	7 5 7 7	
$'6\uparrow_6$	6 2 1	
	5 6 8	
$'5\uparrow_7$	5 3	
	4 7	
$'6\uparrow_8$	6	

∴ '0'5'4'1'8'6'5'6↑ = ·94701510,

which is written '0'5'4'1'8'6'5'6↑ for the sake of convenience.

It is easily shown by the ascending system that

$$947{\cdot}0151 = 2^3(10)^2 \downarrow 1,7,3,7,4,2,3,5,$$

$$\therefore \quad {}'0'5'4'1'8'6'5'6 \uparrow (10)^3 = 2^3(10)^2 \downarrow 1,7,3,7,4,2,3,5,$$

$$\therefore \quad {}'0'5'4'1'8'6'5'6 \uparrow (10) = 2^3 \downarrow 1,7,3,7,4,2,3,5,$$

$$= 2^3 \downarrow 1,7,3,7,4,2,3,5;$$

may be represented by

$$2^3\{1 ⤈ [1 ⤈ [7 ⤈ [3 ⤈ [7 ⤈ [4 ⤈ [2 ⤈ [3 ⤈ [5\}\ ;$$

and

$$0,5,4,1,8,6,5,6, \uparrow (10)$$

is represented thus

$$10\{1 ⤉ [0 ⤉ [5 ⤉ [4 ⤉ [1 ⤉ [8 ⤉ [6 ⤉ [5 ⤉ [6\}$$

⤈ being the sign used in the ascending branch,

and

⤉ the sign employed in the descending.

$$\therefore \quad 4\{1 ⤈ [1 ⤈ [7 ⤈ [3 ⤈ \ldots\} = 5\{1 ⤉ [0 ⤉ [5 ⤉ [4 ⤉ \ldots\}$$

4. Find the dual number and logarithm of 179·170165 by the descending method.

To prepare a number to be operated upon by the descending process, it is necessary to reduce the given number to a decimal fraction; the nearer it is brought to ·99 .... the more readily are its dual representatives found. The necessary preliminary reductions are easily effected by the use of the numbers 10 and 2.

$$179{\cdot}170165 = (2)\,(10)^2\,({\cdot}895850825)$$

$$\begin{array}{rl} {\cdot}895850825 = & {}'1'0'5'\overline{4}'1'8'3'7 \uparrow \\ & 536052 \\ & 250 \\ \hline & 10998139, \text{ negative, written } {}'10998139 \end{array}$$

$$
\begin{array}{lr}
\downarrow,(10^2) & 460517018, \\
\downarrow,(2) & 69314718, \\
\hline
 & +\ 529831736, \\
\downarrow,(\cdot 895850825) = & '10998139 \\
\hline
 & 518833597, = \downarrow,(179\cdot 170165)
\end{array}
$$

Reduction.

$$
\begin{array}{lrl}
 & \cdot 895\,8\,5\,0\,8\,2\,5 & \text{number to be reduced.} \\
\hline
'1'0\,..\uparrow = & \cdot 900|0\,0\,0|0\,0\,0\ + & \\
 & 4|5\,0\,0|0\,0\,0\ - & \\
'5\uparrow_{3} & 9|0\,0\,0\ + & \\
 & 9\ - & \text{diff.} \\
\hline
 & 895\,5|0\,8\,9\,9|1\ + & \\
'\bar{4}\uparrow_{4} & 3|5\,8\,2\,0|3\ + & \\
 & 9|0\ + & \\
\hline
 & 895\,8\,6\,7\,2\,8\,4 & \\
'1\uparrow_{5} & 1|6\,4\,5\,9 & \\
 & |8\,9\,5\,9 & \\
\hline
'8\uparrow_{6} & 7|5\,0\,0 & \\
 & 7|1\,6\,7 & \\
\hline
'3\uparrow_{7} & 3|3\,3 & \\
 & 2|6\,9 & \\
\hline
'7\uparrow_{8} & 6|4 & \\
 & 6|3 & \\
\hline
\end{array}
$$

In the dual number $'1'0'5'\bar{4}'1'8'3'7\uparrow$ one of the digits $'\bar{4}\uparrow_{4}$ is negative, the operating numbers for $'\bar{4}$ in the descending process are $+\ 1$; $+\ 4$; $+\ 10$; $+\ 20$; &c.

In the ascending branch, the operating numbers for $\bar{4}$, are $+1$; -4; $+10$; -20; &c.

$$'1'0'5'\bar{4}'1'8'3'7\uparrow = \{1 \dagger [1 \dagger [0 \dagger [5 \leftarrow [4 \dagger \ldots\ldots \} \text{ descending branch.}$$

$$\downarrow 2,3,\bar{5},1, = \{1 \ddagger [2 \ddagger [3 \rightarrow [5 \ddagger \ldots \} \text{ ascending branch.}$$

$\rightarrow$ being the negative sign in the ascending branch.

and

$\leftarrow$ the negative sign employed in the descending branch.

$\downarrow$; $\ddagger$; $\rightarrow$; ascending signs.

$\uparrow$; $\dagger$; $\leftarrow$; descending signs.

In ascending developments the natural numbers continually increase or ascend, while in descending developments the natural numbers continually decrease or descend; but in both branches the arrows point to the greater number.

$$179{\cdot}170165 = {}'1'0'5'\bar{4}'1'8'3'7\uparrow (10^2)(2)$$

$$= (10^2)(2)\{1 \dagger [1 \dagger [0 \dagger [5 \leftarrow [4 \dagger [1 \dagger [8 \dagger [3 \dagger [7\}$$

$$= \overset{8}{\downarrow} 518833597,$$

and

$$\downarrow, (179{\cdot}170165) = +518833579,$$

or dual log of

$$(179{\cdot}170165) = 518833579,$$

The operating numbers or binomial coefficients for both the ascending and descending processes, and for both positive and negative dual digits may be determined and registered in the following convenient tabulated form. (See page 120.)

	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9
1	3	6	10	15	21	28	36	45
1	4	10	20	35	56	84	120	165
1	5	15	35	70	126	210	330	495
1	6	21	56	126	252	462	792	1287
1	7	28	84	210	462	924	1716	3003
1	8	36	120	330	792	1716	3432	6435
1	9	45	165	495	1287	3003	6435	12870
1								

When the perpendicular and horizontal lines of units are set down, the other numbers are found by simply adding diagonally.

For a dual digit, as 4; on the fourth diagonal line the operating numbers 1; 4; 6; 4; 1 are found, and on the fourth horizontal or perpendicular line the operating numbers 1; 4; 10; 20; 35; &c. are found. These numbers, with the proper signs, are employed in both the ascending and descending operations for the dual digit 4 whether it be positive or negative. The operating numbers for any other dual digit from 1 to 9 are found in the same way in the above table.

Ascending branch.

For the dual digit 4

$$\left\{\begin{array}{llllll} \text{When positive} & +1; & +4; & +6; & +4; & +1 \\ \text{When negative} & +1; & -4; & +10; & +20; & -\ \&c. \end{array}\right\}$$

Descending branch.

For the dual digit 4

$$\left\{\begin{array}{llllll} \text{When positive} & +1; & -4; & +6; & -4; & +1 \\ \text{When negative} & +1; & +4; & +10; & +20; & +\ \&c. \end{array}\right\}$$

$$\overset{1}{\downarrow} 4, = 1\cdot464100000$$

$$\overset{1}{\downarrow} \bar{4}, = \cdot6830134555$$

$$'4 \underset{1}{\uparrow} = \cdot656100000$$

$$'\bar{4} \underset{1}{\uparrow} = 1\cdot524157503$$

DUAL DEVELOPMENTS BY THE APPLICATION OF BOTH THE ASCENDING AND DESCENDING BRANCHES OF THE ART COMBINED.

$$N = {}'v_1{}'v_2{}'v_3 \ldots {}_n\updownarrow{}_m\, u_1, u_2, u_3, \ldots = \overset{8}{\downarrow} n,$$

or,

$$N = {}'v'v'v \ldots {}_n\updownarrow{}_m\, u,u,u, \ldots = \overset{8}{\downarrow} n,$$

Dual log of $N = n$, is written

$$\downarrow,(N) = n,$$

$\overset{5}{\downarrow} u$, may be written $\downarrow u_5$, or $\downarrow 0,0,0,0,u$,

$v \underset{3}{\uparrow}$ may be written $'v_3 \ldots \uparrow$ or $'0'0'v \ldots \uparrow$

m represents some power of 10 and n some power of 2.

It may be observed that a dual development has TWO branches, ($'v'v'v \ldots \uparrow$) and ($\downarrow u,u,u, \ldots$); A DOUBLE sign ($\updownarrow$); TWO ultimate values, the natural number (N) and its logarithm (n,) to a known base; and TWO powers (m and n) of TWO simple numbers (10 and 2); hence the art has a good claim to the title Dual Arithmetic.

To find the dual log of 3· and 9·

$$10 = 9 \downarrow 1,1,0,1,0,0,0,1, = 9 \overset{8}{\downarrow} 10536052,$$

$$\downarrow,(10) = 230258509,$$
$$10536052,$$

$$\downarrow,(9) = 219722457,$$

$$\downarrow,(3) = 109861229,$$

1. **Find the dual logarithm of 3·1415926535 and give all the ures employed in the operation.**

$$\begin{array}{r} 3{\cdot}1415926535 \\ 3 \\ \hline 9{\cdot}42477796\ldots \end{array}$$

$$\begin{array}{rrl}
 & {\cdot}942\,47\,7\,7\,96 & \\
\hline\hline
'0'6\ldots\uparrow = & {\cdot}941|48\,0|1\,49 & \overset{3}{\downarrow}\ 1, \\
 & 94\,1|4\,80 & \\
\hline
 & 942\,42\,1\,6\,29 & \\
 & 5|6\,1\,67 & \overset{5}{\downarrow}\ 5, \\
 & 4|7\,1\,21 & \\
\hline
 & 9|0\,46 & \overset{6}{\downarrow}\ 9, \\
 & 8|4\,82 & \\
\hline
 & 5|64 & \overset{7}{\downarrow}\ 6, \\
 & 5|65 & \\
\hline
\end{array}$$

$$\therefore\quad 3{\cdot}14159265\ldots = '0'6'0'0'0'0'0'0\uparrow\tfrac{10}{3}\downarrow 0{,}0{,}1{,}0{,}5{,}9{,}6{,}0,$$

$$\begin{array}{rr}
30204 & 50 \\
\hline
'6030204 & 105910, \\
105910, & \\
\hline
\text{negative}\quad '5924294 &
\end{array}$$

$$\begin{array}{rl}
\downarrow,(10) = & 230258509, \\
\downarrow,(3) = & 109861229, \\
\hline
 & 120397280, \\
 & '5924294 \\
\hline
\therefore\quad \downarrow,(\pi) = & 114472986,
\end{array}$$

$\pi = \frac{10}{3}\{1 \underset{2}{\uparrow} [6 \overset{3}{\downarrow} [1 \overset{5}{\downarrow} [5 \overset{6}{\downarrow} [9 \overset{7}{\downarrow} [6\}$ in which but five digits are employed, one belonging to the descending branch, namely, (6)' and four belonging to the ascending branch, (1,5,9,6,). By comparing developments by each branch separately, and with both branches combined, the advantages to be gained by the combined operation will be readily perceived. Yet, in particular developments, the application of the ascending or descending branch alone will be found most convenient.

Descending.

$$1\cdot 11111111 = {}'\bar{1}'0'0'0'0'0'0'0 \uparrow = -\ 10536052 \underset{8}{\uparrow}$$

$$\therefore \quad \downarrow, (1\cdot 11111111) = +\ 10536052,$$

Ascending.

$$1\cdot 11111111 = \downarrow 1,1,0,1,0,0,0,1, = \downarrow 10536052,$$

Descending.

$$1\cdot 23456789 = {}'\bar{2}'0'0'0'0'0'0'0 \uparrow = -\ 21072104 \underset{8}{\uparrow}$$

$$\therefore \quad \downarrow, (1\cdot 23456789) = +\ 21072104,$$

Descending.

$$1\cdot 37174211 = {}'\bar{3}'0'0'0'0'0'0'0 \uparrow = -\ 31608156 \underset{8}{\uparrow}$$

$$\therefore \quad \downarrow, (1\cdot 37174211) = +\ 31608156,$$

2. Find the dual logarithm and dual number corresponding to $\cdot 765432110$.

BY BOTH BRANCHES COMBINED.

```
            ·765432110  given number
           ============
'3 ↑ = ·72|90|00|00|0
    1     3|64|50|00|0
              7|29|00|0        2
                 7|29|0        ↓ 5,
                    3|6
           ------------
'1 ↑      766|186|326
              |766|186
           ------------
          765420140
               11970           5
                7654           ↓ 1,
               -----
                4316           6
                3827           ↓ 5,
               -----
                 489           7
                 465           ↓ 6,
               -----
                  24           8
                  23           ↓ 3,
               -----
```

∴ ·76543211 = '3'0'1'0'0'0'0'0 ↑ 0,5,0,0,1,5,6,3, = '26731478 ↑

Reduction to a Dual Logarithm.

See (12), (13), (14).

```
+ '3'0'1'0'0'0'0'0 ↑ 0,5,0,0,1,5,6,3, +
+   150005              25000 −
+    108156               165 +
  ------------      -----------
 '31708206          4976728,
   4976728,
  ------------
 '26731478
```

∴ ↓,(·76543211) = '26731478

Reduction by the Ascending Branch.

7) ·765432110

Number to be reduced 1·09347444 $= \frac{7}{10}(1{\cdot}09347444)$

$\overset{2}{\downarrow} 9, =$ 1093 | 6852 | 7 + $\qquad \overset{4}{\downarrow} \bar{2},$

| 2187 | 4 −

| 3 +

1093 46656

788 $\qquad \overset{6}{\downarrow} 7,$

765

23 $\qquad \overset{7}{\downarrow} 2,$

22 $\qquad \overset{8}{\downarrow} 0,$

$\therefore \; {\cdot}76543211 = \frac{7}{10} \downarrow 0,9,0,\bar{2},0,7,2,0, = \overset{8}{\downarrow} - 26731476,$

See Rule, Article (12).

297 + nine times 33

$\downarrow$ 0,9,0,$\bar{2}$,0,7,2,0,

45000 − subtract.

89360017

$\downarrow$,(7) = 194591016

203527033 +

$\downarrow$,(10) = 230258509 −

26731476 − or ’26731476

Reduction by the Descending Branch.

	·765 432 110	given number.
'2'5 . . $\uparrow$ =	·770|301|940	+
	4|621|812	−
'6 $\uparrow_8$	11|557	+
	15	−
	765 6|9 1 67|0	+
'3 $\uparrow_4$	2|2970|8	−
	2 3	+
	765 4 61 985	
'3 $\uparrow_5$	2|9875	
	2|2974	
'9 $\uparrow_6$	6|901	
	6|889	
'2 $\uparrow_8$	1|2	

∴ ·76543211 = '2'5'6'3'3'9'0'2 $\uparrow$ = '26731476 $\uparrow$

and

∴ $\downarrow$, (·76543211) = '26731476

Reduction by the Rule, page 19.

'2'5'6'3'3'9'0'2 $\uparrow$		
1025300	5 times	205060
72104	2 times	36052
170	5 times	34
'26731476		

Reductions may be effected in a great number of ways by both branches of the art, jointly or severally applied. Every method is continuous without interruption, since no inconvenience is experienced by intermediate results becoming too

great or too small within proper limits. However, the best method to be employed in each particular case must be left to the skill or design of the operator.

$$\cdot 76543211 = \frac{7}{10}\{1 \overset{2}{\downarrow} [9 \overset{4}{\longrightarrow} [2 \overset{6}{\downarrow} [7 \overset{7}{\downarrow} [2\}$$

$$= \{1 \uparrow [2 \uparrow [5 \uparrow [6 \uparrow [3 \uparrow [3 \uparrow [9 \underset{8}{\uparrow} [2\}$$

$$= \{1 \underset{1}{\uparrow} [3 \underset{3}{\uparrow} [1 \overset{2}{\downarrow} [5 \overset{5}{\downarrow} [1 \overset{6}{\downarrow} [5 \overset{7}{\downarrow} [6 \overset{8}{\downarrow} [3\}$$

In general terms,

$$'\mathrm{O}'v_2'\mathrm{O}'\bar{v}_4 \ldots \updownarrow u_1, \bar{u}_3, u_5, \ldots = \{1 \downarrow [u_1 \uparrow [v_2 \longrightarrow [u_3 \longleftarrow [v_4 \uparrow [u_5\}$$

$$'v_1'\mathrm{O}'\bar{v}_3'\mathrm{O}'v_5 \ldots \left\uparrow \frac{r}{s} \right\downarrow \mathrm{O}, \bar{u}_2, \mathrm{O}, u_4, \mathrm{O}, u_6, \ldots$$

$$= \frac{r}{s}\{1 \uparrow [v_1 \longrightarrow [u_2 \longleftarrow [v_3 \downarrow [u_4 \uparrow [v_5 \downarrow [u_6\}$$

$$= \frac{r}{s}\{1 \uparrow [v_1 \longrightarrow [u_2\} \{1 \longleftarrow [v_3 \downarrow [u_4 \uparrow [v_5\} \{1 \downarrow [u_6\}$$

&c. &c. &c.

From the foregoing practical applications of the dual signs, it is presumed that their functions, or the operations indicated by them, will be understood.

$$\text{Dual signs} \left\{ \begin{array}{ll} \downarrow \quad \downarrow \quad \longrightarrow \quad \downarrow & \text{ascending.} \\ \updownarrow & \text{both branches combined.} \\ \uparrow \quad \uparrow \quad \longleftarrow \quad \uparrow & \text{descending.} \end{array} \right.$$

A dual number reduced to the eight position, on account of its practical importance, is termed a dual logarithm. In practice it is seldom necessary to use more than eight dual digits, or a dual number reduced to a higher position than the eight, since results are rarely required to be true beyond the seventh decimal place. However, a system may be soon framed to secure any required degree of accuracy; as an example of such extensions,

dual numbers to the seventeenth position are employed in the following developments.

$|1, \quad = \downarrow 0,9,5,7,5,9,7,3,5,7,1,2,5,9,4,7,7, = \overset{17}{\downarrow} 9531017980432486,016$

$|0,1, \quad = \downarrow 0,0,9,9,5,4,8,7,3,1,0,4,4,5,5,2,0, = \overset{17}{\downarrow} 9950330853168 08,288$

$|0,0,1, \quad = \downarrow 0,0,0,9,9,9,5,4,5,7,8,4,6,0,5,9,6, = \overset{17}{\downarrow} 99950033308353,350$

$|0,0,0,1, \quad = \downarrow 0,0,0,0,9,9,9,9,5,4,5,4,8,7,5,7,6, = \overset{17}{\downarrow} 9999500033330,833$

$\downarrow 0,0,0,0,1 \quad = \downarrow 0,0,0,0,0,9,9,9,9,9,5,4,5,4,5,8,0, = \overset{17}{\downarrow} 999995000033,333$

$\downarrow 0,0,0,0,0,1, \quad = \downarrow 0,0,0,0,0,0,9,9,9,9,9,9,5,4,5,4,5, = \overset{17}{\downarrow} 99999950000,033$

$\downarrow 0,0,0,0,0,0,1, = \downarrow 0,0,0,0,0,0,0,9,9,9,9,9,9,9,5,4,5, = \overset{17}{\downarrow} 9999999500,$

$\downarrow 0,0,0,0,0,0,0,1, = \downarrow 0,0,0,0,0,0,0,0,9,9,9,9,9,9,9,9,5, = \overset{17}{\downarrow} 999999995,$

The work of one of these reductions will show how the above results are obtained.

```
 5
↓9,   10000|00000|00000|000
           |90000|00000|000
                3|6000 0|000
                       8|400
      ----------------------
 6
↓9,   100009|000360|008400
            |900081|003240
                  3|600324
                           8
      ----------------------
 7
↓9,   1000099|0044461|1972
             |9000891|0400
                    3|6004
      ----------------------
 8
↓9,   10000999|04535583|76
              |90008991|41
                      3|60
      ----------------------
 9
↓5,   100009999|4544578 77
               |500049997
                         1
      ----------------------
      10000999995450787 5
```

10000 99999 54 50 7 8 75

$\overset{10}{\downarrow}4$ 45 49 2 1 25
40 00 4 0 00

$\overset{11}{\downarrow}5,$ 5 48 8 1 25
5 000 5 00

$\overset{12}{\downarrow}4,$ 48 7 6 25
400040,

$\overset{13}{\downarrow}8,$ 8 7 5 85
80008

$\overset{14}{\downarrow}7,$ 7 5 77
7001

$\overset{15}{\downarrow}$ 5,7,6,

∴ 1·0001 = $\overset{4}{\downarrow}$ 1, = $\downarrow$ 0,0,0,0,9,9,9,9,5,4,5,4,8,7,5,7,6,

(See "Dual Arithmetic, a New Art," page 42, and "The Young Dual Arithmetician," pp. 72 to 80.)

The dual numbers $\overset{1}{\downarrow}$ 1, ; $\overset{2}{\downarrow}$ 1, ; $\overset{3}{\downarrow}$ 1, ; &c. are reduced to the seventeenth position as follows.

$\downarrow$ 0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0, = $\overset{17}{\downarrow}$ 999999995, = $\overset{8}{\downarrow}$ 1,

Mult. by 9 $\downarrow$ 0,0,0,0,0,0,0,9,0,0,0,0,0,0,0,0,0, = $\overset{17}{\downarrow}$ 8999999955, (*a*)
Add 9,9,9,9,9,9,5,4,5, 999999545,

$\downarrow$ 0,0,0,0,0,0,0,9,9,9,9,9,9,5,4,5, = $\overset{17}{\downarrow}$ 9999999500, = $\overset{7}{\downarrow}$ 1,

∴ $\downarrow$ 0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0, = $\overset{17}{\downarrow}$ 9999999500, = $\overset{7}{\downarrow}$ 1,

Mult. by 9 $\downarrow$ 0,0,0,0,0,0,9,0,0,0,0,0,0,0,0,0,0, = $\overset{17}{\downarrow}$ 89999995500, (*b*)
9,0,0,0,0,0,0,0,0,0,0, = $\overset{17}{\downarrow}$ 8999999955, (*a*)
Add 9,9,9,9,5,4,5,4,5, 999954545,

$\downarrow$ 0,0,0,0,0,0,9,9,9,9,9,9,5,4,5,4,5, = $\overset{17}{\downarrow}$ 99999950000, = $\overset{6}{\downarrow}$ 1,

$\therefore\quad \downarrow 0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0, = \overset{17}{\downarrow}\ 99999950000, = \overset{6}{\downarrow} 1,$

Mult. by 9 $\left.\right\}\downarrow 0,0,0,0,0,9,0,0,0,0,0,0,0,0,0,0,0, = \overset{17}{\downarrow}\ 899999550000, \quad (c)$

$9,0,0,0,0,0,0,0,0,0,0, = \quad 89999995500, \quad (b)$

$9,0,0,0,0,0,0,0,0,0, = \quad 8999999955, \quad (a)$

Add $\quad 9,9,5,4,5,4,5,7,9, = \quad 995454579,$

$\downarrow 0,0,0,0,0,9,9,9,9,9,5,4,5,4,5,7,9, = \overset{17}{\downarrow}\ 999995000033,$

$\therefore\quad \downarrow 0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0, = \overset{17}{\downarrow}\ 999995000033, = \overset{5}{\downarrow} 1,$

Mult. by 9 $\left.\right\}\downarrow 0,0,0,0,9,0,0,0,0,0,0,0,0,0,0,0,0, = \overset{17}{\downarrow}\ 8999955000300, \quad (d)$

$9,0,0,0,0,0,0,0,0,0,0,0, = \quad 899999550000, \quad (c)$

$9,0,0,0,0,0,0,0,0,0,0, = \quad 89999995500, \quad (b)$

$9,0,0,0,0,0,0,0,0,0, = \quad 8999999955, \quad (a)$

Add $\quad 5,4,5,4,8,7,5,7,6, = \quad 545487576,$

$\downarrow 0,0,0,0,9,9,9,9,5,4,5,4,8,7,5,7,6, = \overset{17}{\downarrow}\ 9999500033331,$

$\therefore\quad \downarrow 0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0, = \overset{17}{\downarrow}\ 9999500033331, = \overset{4}{\downarrow} 1,$

Mult. by 9 $\left.\right\}\downarrow 0,0,0,9,0,0,0,0,0,0,0,0,0,0,0,0,0, = \overset{17}{\downarrow}\ 89995500299977, \quad (e)$

$9,0,0,0,0,0,0,0,0,0,0,0,0, = \quad 8999955000300, \quad (d)$

$9,0,0,0,0,0,0,0,0,0,0,0, = \quad 899999550000, \quad (c)$

$5,0,0,0,0,0,0,0,0,0,0, = \quad 49999997500, \quad \overset{7}{\downarrow} 5,$

$4,0,0,0,0,0,0,0,0,0, = \quad 3999999980, \quad \overset{8}{\downarrow} 4,$

Add $\quad 5,7,8,4,6,0,5,9,6, = \quad 578460596,$

$\downarrow 0,0,0,9,9,9,5,4,5,\dot{7},8,4,6,0,5,9,6, = \overset{17}{\downarrow}\ 99950033308353,$

$\therefore\ \downarrow 0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0, = \overset{17}{\downarrow}\ 99950033308353, = \overset{8}{\downarrow} 1,$

Mult. by 9 $\Big\}\ \downarrow 0,0,9,0,0,0,0,0,0,0,0,0,0,0,0,0,0, = \overset{17}{\downarrow}\ 899550299775180,\ (f)$

$9,0,0,0,0,0,0,0,0,0,0,0,0,0, = \ 89995500299977,\ (e)$

$5,0,0,0,0,0,0,0,0,0,0,0,0, = \ 4999975000167,\ \overset{5}{\downarrow}5,$

$4,0,0,0,0,0,0,0,0,0,0,0, = \ 399999800000,\ \overset{6}{\downarrow}4,$

$8,0,0,0,0,0,0,0,0,0,0, = \ 79999996000,\ \overset{7}{\downarrow}8,$

$7,0,0,0,0,0,0,0,0,0, = \ 6999999965,\ \overset{8}{\downarrow}7,$

Add $3,1,0,4,4,5,5,2,0, = \ 310445520,$

$\downarrow 0,0,9,9,5,4,8,7,3,1,0,4,4,5,5,2,0, = \overset{17}{\downarrow}\ 995033085316809,$

$\therefore\ \downarrow 0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0, = \overset{17}{\downarrow}\ 995033085316809, = \overset{2}{\downarrow} 1,$

Mult. by 9 $\Big\}\ \downarrow 0,9,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0, = \overset{17}{\downarrow}\ 8955297767851275,\ \overset{2}{\downarrow}9,$

$5,0,0,0,0,0,0,0,0,0,0,0,0,0,0, = \ 499750166541767,\ \overset{3}{\downarrow}5,$

$7,0,0,0,0,0,0,0,0,0,0,0,0,0, = \ 69996500233316,\ \overset{4}{\downarrow}7,$

$5,0,0,0,0,0,0,0,0,0,0,0,0, = \ 4999975000167,\ \overset{5}{\downarrow}5,$

$9,0,0,0,0,0,0,0,0,0,0,0, = \ 899999550000,\ \overset{6}{\downarrow}9,$

$7,0,0,0,0,0,0,0,0,0,0, = \ 69999996500,\ \overset{7}{\downarrow}7,$

$3,0,0,0,0,0,0,0,0,0, = \ 2999999985,\ \overset{8}{\downarrow}3,$

Add $5,7,1,2,5,9,4,7,7, = \ 571259477,$

$\downarrow 0,9,5,7,5,9,7,3,5,7,1,2,5,9,4,7,7, = \overset{17}{\downarrow}\ 9531017980432487,$

$\therefore\ \downarrow 1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0, = \overset{17}{\downarrow}\ 9531017980432486,$

The numbers here registered are true to the last figure; to secure the designed degree of accuracy, the calculations were made for dual numbers of twenty digits, and when broken off to seventeen, the proper allowances were made. The same degree of accuracy is established in the following tabulated multiples in a similar manner.

The values of $'1\uparrow$; $'0'1\uparrow$; $'0'0'1\uparrow$; &c., in the seventeenth position were found as follows.

Since $\cdot 9 \downarrow 1,1,0,1,0,0,0,1,0,0,0,0,0,0,0,1,0, = 1.$

$$\therefore \overset{1}{\downarrow} 1, = \overset{17}{\downarrow} 95310179804324860{,}016$$

$$\overset{2}{\downarrow} 1, = 9950330853168083{,}288$$

$$\overset{4}{\downarrow} 1, = 99995000333330{,}833$$

$$\overset{8}{\downarrow} 1, = 999999995,$$

$$\overset{16}{\downarrow} 1, = 10,$$

$$'1\uparrow = '10536051565782630\underset{17}{\uparrow} \quad '5\uparrow = '52680257828913150\underset{17}{\uparrow}$$

$$'2\uparrow = '21072103131565260\underset{17}{\uparrow} \quad '3\uparrow = '31608154697347890\underset{17}{\uparrow}$$

$$'4\uparrow = '42144206263130520\underset{17}{\uparrow} \quad '6\uparrow = '63216309394695780\underset{17}{\uparrow}$$

$$'8\uparrow = '84288412526261040\underset{17}{\uparrow} \quad '9\uparrow = '94824464092043670\underset{17}{\uparrow}$$

$$10536051565782630$$

$$'7\uparrow = '73752360960478410\underset{17}{\uparrow}$$

	$\overset{1}{\downarrow}$ 1,
1	$\overset{17}{\downarrow}$ 9531017980432486,
2	$\overset{17}{\downarrow}$ 19062035960864972,
3	$\overset{17}{\downarrow}$ 28593053941297458,
4	$\overset{17}{\downarrow}$ 38124071921729944,
5	$\overset{17}{\downarrow}$ 47655089902162430,
6	$\overset{17}{\downarrow}$ 57186107882594916,
7	$\overset{17}{\downarrow}$ 66717125863027402,
8	$\overset{17}{\downarrow}$ 76248143843459888,
9	$\overset{17}{\downarrow}$ 85779161823892374,

	$\overset{3}{\downarrow}$ 1,
1	$\overset{17}{\downarrow}$ 99950033308353,
2	$\overset{17}{\downarrow}$ 199900066616707,
3	$\overset{17}{\downarrow}$ 299850099925060,
4	$\overset{17}{\downarrow}$ 399800133233413,
5	$\overset{17}{\downarrow}$ 499750166541767,
6	$\overset{17}{\downarrow}$ 599700199850120,
7	$\overset{17}{\downarrow}$ 699650233158473,
8	$\overset{17}{\downarrow}$ 799600266466827,
9	$\overset{17}{\downarrow}$ 899550299775180,

	$\overset{2}{\downarrow}$ 1,
1	$\overset{17}{\downarrow}$ 995033085316808,
2	$\overset{17}{\downarrow}$ 1990066170633616,
3	$\overset{17}{\downarrow}$ 2985099255950425,
4	$\overset{17}{\downarrow}$ 3980132341267233,
5	$\overset{17}{\downarrow}$ 4975165426584041,
6	$\overset{17}{\downarrow}$ 5970198511900850,
7	$\overset{17}{\downarrow}$ 6965231597217658,
8	$\overset{17}{\downarrow}$ 7960264682534466,
9	$\overset{17}{\downarrow}$ 8955297767851275,

	$\overset{4}{\downarrow}$ 1,
1	$\overset{17}{\downarrow}$ 99995000333331,
2	$\overset{17}{\downarrow}$ 199990000666662,
3	$\overset{17}{\downarrow}$ 299985000999993,
4	$\overset{17}{\downarrow}$ 399980001333323,
5	$\overset{17}{\downarrow}$ 499975001666654,
6	$\overset{17}{\downarrow}$ 599970001999985,
7	$\overset{17}{\downarrow}$ 699965002333316,
8	$\overset{17}{\downarrow}$ 799960002666647,
9	$\overset{17}{\downarrow}$ 899955002999977,

	$\overset{5}{\downarrow}$ I,
I	$\overset{17}{\downarrow}$ 999995000033,
2	$\overset{17}{\downarrow}$ 1999990000067,
3	$\overset{17}{\downarrow}$ 2999985000100,
4	$\overset{17}{\downarrow}$ 3999980000133,
5	$\overset{17}{\downarrow}$ 4999975000167,
6	$\overset{17}{\downarrow}$ 5999970000200,
7	$\overset{17}{\downarrow}$ 6999965000233,
8	$\overset{17}{\downarrow}$ 7999960000267,
9	$\overset{17}{\downarrow}$ 8999955000300,

	$\overset{6}{\downarrow}$ I,
I	$\overset{17}{\downarrow}$ 99999950000,
2	$\overset{17}{\downarrow}$ 199999900000,
3	$\overset{17}{\downarrow}$ 299999850000,
4	$\overset{17}{\downarrow}$ 399999800000,
5	$\overset{17}{\downarrow}$ 499999750000,
6	$\overset{17}{\downarrow}$ 599999700000,
7	$\overset{17}{\downarrow}$ 699999650000,
8	$\overset{17}{\downarrow}$ 799999600000,
9	$\overset{17}{\downarrow}$ 899999550000,

	$\overset{7}{\downarrow}$ I,
I	$\overset{17}{\downarrow}$ 9999999500,
2	$\overset{17}{\downarrow}$ 19999999000,
3	$\overset{17}{\downarrow}$ 29999998500,
4	$\overset{17}{\downarrow}$ 39999998000,
5	$\overset{17}{\downarrow}$ 49999997500,
6	$\overset{17}{\downarrow}$ 59999997000,
7	$\overset{17}{\downarrow}$ 69999996500,
8	$\overset{17}{\downarrow}$ 79999996000,
9	$\downarrow$ 89999995500,

	$\overset{8}{\downarrow}$ I,
I	$\overset{17}{\downarrow}$ 999999995,
2	$\overset{17}{\downarrow}$ 1999999990,
3	$\overset{17}{\downarrow}$ 2999999985,
4	$\overset{17}{\downarrow}$ 3999999980,
5	$\overset{17}{\downarrow}$ 4999999975,
6	$\overset{17}{\downarrow}$ 5999999970,
7	$\overset{17}{\downarrow}$ 6999999965,
8	$\overset{17}{\downarrow}$ 7999999960,
9	$\overset{17}{\downarrow}$ 8999999955,

$'1_{1}\uparrow$ or, $'1\uparrow$	
$'10536051565782630_{17}\uparrow$	1
$'21072103131565260_{17}\uparrow$	2
$'31608154697347890_{17}\uparrow$	3
$'42144206263130520_{17}\uparrow$	4
$'52680257828913150_{17}\uparrow$	5
$'63216309394695780_{17}\uparrow$	6
$'73752360960478410_{17}\uparrow$	7
$'84288412526261040_{17}\uparrow$	8
$'94824464092043670_{17}\uparrow$	9

$'1_{3}\uparrow$ or, $'0'0'1\uparrow$	
$'100050033358353_{17}\uparrow$	1
$'200100066716706_{17}\uparrow$	2
$'300150100075059_{17}\uparrow$	3
$'400200133433412_{17}\uparrow$	4
$'500250166791765_{17}\uparrow$	5
$'600300200150118_{17}\uparrow$	6
$'700350233508471_{17}\uparrow$	7
$'800400266866824_{17}\uparrow$	8
$'900450300225177_{17}\uparrow$	9

$'1_{2}\uparrow$ or, $'0'1\uparrow$	
$'10050335853350144_{17}\uparrow$	1
$'2010067170700288_{17}\uparrow$	2
$'3015100756050432_{17}\uparrow$	3
$'4020134341400576_{17}\uparrow$	4
$'5025167926750720_{17}\uparrow$	5
$'6030201512100864_{17}\uparrow$	6
$'7035235097451008_{17}\uparrow$	7
$'8040268682801152_{17}\uparrow$	8
$'9045302268151296_{17}\uparrow$	9

$'1_{4}\uparrow$ or, $'0'0'0'1\uparrow$	
$'10000500033336_{17}\uparrow$	1
$'20001000066672_{17}\uparrow$	2
$'30001500100008_{17}\uparrow$	3
$'40002000133344_{17}\uparrow$	4
$'50002500166680_{17}\uparrow$	5
$'60003000200016_{17}\uparrow$	6
$'70003500233352_{17}\uparrow$	7
$'80004000266688_{17}\uparrow$	8
$'90004500300024_{17}\uparrow$	9

$'1\uparrow_{5}$ or, $'0'0'0'0'1\uparrow$	
$'1000005000033\uparrow_{17}$	1
$'2000010000066\uparrow_{17}$	2
$'3000015000099\uparrow_{17}$	3
$'4000020000132\uparrow_{17}$	4
$'5000025000165\uparrow_{17}$	5
$'6000030000192\uparrow_{17}$	6
$'7000035000231\uparrow_{17}$	7
$'8000040000264\uparrow_{17}$	8
$'9000045000297\uparrow_{17}$	9

$'1\uparrow_{7}$	
$'10000000500\uparrow_{17}$	1
$'20000001000\uparrow_{17}$	2
$'30000001500\uparrow_{17}$	3
$'40000002000\uparrow_{17}$	4
$'50000002500\uparrow_{17}$	5
$'60000003000\uparrow_{17}$	6
$'70000003500\uparrow_{17}$	7
$'80000004000\uparrow_{17}$	8
$'90000004500\uparrow_{17}$	9

$'1\uparrow_{6}$	
$'100000050000\uparrow_{17}$	1
$'200000100000\uparrow_{17}$	2
$'300000150000\uparrow_{17}$	3
$'400000200000\uparrow_{17}$	4
$'500000250000\uparrow_{17}$	5
$'600000300000\uparrow_{17}$	6
$'700000350000\uparrow_{17}$	7
$'800000400000\uparrow_{17}$	8
$'900000450000\uparrow_{17}$	9

$'1\uparrow_{8}$	
$'1000000005\uparrow_{17}$	1
$'2000000010\uparrow_{17}$	2
$'3000000015\uparrow_{17}$	3
$'4000000020\uparrow_{17}$	4
$'5000000025\uparrow_{17}$	5
$'6000000030\uparrow_{17}$	6
$'7000000035\uparrow_{17}$	7
$'8000000040\uparrow_{17}$	8
$'9000000045\uparrow_{17}$	9

In most cases, reduction by both branches of the art combined are far more easily effected than by the application of either branch employed alone, as the following examples will tend to show.

1. Reduce 2 to a dual number and to a dual logarithm in the seventeenth position.

```
           21|43|5888|10|00|00|0000 + = ↓ 8,
            1|50|05  2|16|70|00|00000 −
'7 ↑          4|50  5|36|50|10|0000 +
   2             750|26|60|83|5000 −
                    7|50|25|60|8350 +
                       4|50|15|3650 −
                          1|50|05|12 +
                                21|44 −
           ------------------------------
           199|796|484|996|218|068          3
             |199|796|484|996|2'18         ↓ 1,
           ------------------------------
           19999|62814|81214286
                |39999|25629624             5
                      |19999628            ↓ 2,
           ------------------------------
'1 ↑       200000|281426843538
   6             |200000281427
           ------------------------------
'4 ↑       2000000|814265|62111 +
   7              |8000003|2571 −
                          1|2000 +.
           ------------------------------
'7 ↑       2000000001|426541540 +
   9                 1|400000010 −
                                4 +
           ------------------------------
           2000000000|26541534
                     |--------
                     |1,3,2,7,0,7,6,7,
```

$$'0'7'0'0'0'1'4'0|'7'1'3'2'7'0'7'6'7 \updownarrow 8,0,1,0,2,0,0,0,0,0,0,0,0,0,0,0,0, = 2$$

7035235 09745 1008,	76248143843459888,	Taken from the Tables.
100 000050000,	99950033308353,	
40 000002000,	19999990000067,	
703537 58107737775,	76350093866768308,	
	70353758107737775,	
	69314718055994533;	

$$\therefore \quad 2 = \overset{17}{\downarrow} 69314718055994533,$$

Since $\downarrow 8$, may be set down without calculation, only six digits out of the seventeen require attention, namely, $'7\underset{2}{\uparrow}$; $'1\underset{5}{\uparrow}$; $'4\underset{7}{\uparrow}$; $'7\underset{9}{\uparrow}$; $\overset{3}{\downarrow} 1,$; $\overset{5}{\downarrow} 2,$; and two of these are units.

2. Reduce 3, 9, and 10, to a dual number in the seventeenth position.

$$3 \times \tfrac{2}{3} = 2$$

$$\therefore \quad \overset{17}{\downarrow},(3) + \overset{17}{\downarrow},(\tfrac{2}{3}) = \overset{17}{\downarrow},(2)$$

$$\therefore \quad \overset{17}{\downarrow},(3) = \overset{17}{\downarrow},(2) - \overset{17}{\downarrow},(\tfrac{2}{3})$$

$'4\uparrow = \cdot 65|61|$
$|65|61 \qquad \overset{2}{\downarrow} 1,$

662	661	000	000	000	000	00
3	975	966	000	000	000	00
	9	939	915	000	000	00
		13	253	220	000	00
			9	939	915	00
				3	975	97
						66
66664	69191	78163	89163			

$\overset{3}{\downarrow} 6,$

```
         66664|69191|78163|89163
             1|99994|07575|34492        5 ↓ 3,
                   1|99994|07575
                         |66647
         ---------------------------
         6666669|1878573|397877 +
 '4↑ 7         2|6666676|751429 −
                       4|000001 +
                               3 −
         ---------------------------
         66666665|21190064|6446
                1|33333330|4238         8 ↓ 2,
                          |6667
         ---------------------------
         666666665|452339573|51
                 1|333333330|90         9 ↓ 2,
                            |67
         ---------------------------
 '2↑ 10  66666666667,8567290508
                   1|3333333336
         ---------------------------
         666666666665233957172
         =====================
                   1|432709495         11 ↓ 2,
                   1|333333333
                   -----------
                     99376162          12 ↓ 1,
                     66666667
                     --------
                     32709495          13 ↓ 4,
                     26666667
                     --------
                      6042828          14 ↓ 9,
                      6000000
                      -------
                       4|2828          16 ↓ 6,
                       4|0000
                       ------
                       2|828           17 ↓ 4,
```

Contracted operations are avoided in these extended reductions; it may be remarked however, that the remainder 1432709495 multiplied by $\frac{3}{2}$ will also gives the required result. Thus,

```
      1432709|495
               3
      -----------
  2 ) 4298128|485
      -----------
      2,1,4,9,0,6,4,
```

'4'0'0'0'0'0'4'0'0'2'0'0'0'0'0'0'0 ↓ 0,1,6,0,3,0,0,2,2,0,2,1,4,9,0,6,4, = $\frac{2}{3}$

'4'0'0'0'0'0'4'0 | '0'2'0'0'0'0'0'0'0 $\updownarrow$	0,1,6,0,3,0,0,2, | 2,0,2,1,4,9,0,6,4,
42144206 263130520,	995033 085316808, (add)
40 000002000,	599700 199850120, (add)
	2999 985000100, (add)
42144246 263132520,	1 999999990, (add)
1597735 472316082,	1597735 472316082,
405465108108164 38,	

$$\therefore\quad '405465108108164 38 = \left(\tfrac{2}{3}\right)'\underset{17}{\uparrow}$$

$$\overset{17}{\downarrow},(2) = 69314718055994531,$$
$$405465108108164 38, = \overset{17}{\downarrow},\left(\tfrac{3}{2}\right)$$
$$\overset{17}{\downarrow},(3) = 10986122886681096 9,$$
$$2$$
$$\overset{17}{\downarrow},(9) = 21972245773362193 8,$$
$$10536051565782630, = '1\underset{17}{\uparrow}$$
$$\overset{17}{\downarrow},(10) = 23025850929940456 8,$$

3. **Find the hyperbolic logarithm, the common logarithm, and the dual logarithm in the seventeenth position of**

$$\pi = 3{\cdot}14159265358979323846$$

$$\begin{array}{r} 3{\cdot}141\,5926535\,8979\,323846 \\ 3 \\ \hline 9{\cdot}424\,7779607\,6937972 \end{array}$$

$$'6\underset{2}{\uparrow} = \begin{array}{l} \cdot 941\,|\,48\,0\,|\,1\,49\,|\,4\,0\,1\,|\,0\,0\,0\,|\,000 \\ \quad\;\; |\,94\,1\,|\,4\,80\,|\,1\,4\,9\,|\,4\,0\,1\,|\,000 \\ \hline \;942\,42\,|\,1\,6\,29\,5\,|\,5\,0\,4\,0\,1\,|\,000 \end{array} \qquad \overset{3}{\downarrow}\,1,$$

```
942 42|1 6 29 5|5 0 4 0 1|000
     4|7 1 21 0|8 1 4 7 7|520        ↓⁵ 5,
               9|4 2 4 2 1|630
                         9|424
------------------------------
942 46|8 7 5 1 5|7 4 3 0 9|574       ↓⁵ 1,
      |9 4 24 6|8 7 5 1 5|743
------------------------------
942 47 8 1|76 2 6 1 8 2|5 317 +
'4 ↑₇          3|76 9 9 1 2 7|0 505 −
                         5|6 549 +
------------------------------
942 47 7 7 99|2 7 0 6 1 1 361 +
'3 ↑₉              2|8 2 7 4 3 3 398 −
                                3 +
------------------------------
'4 ↑₁₀ 942 47 7 7 96 4|4 3 1 7 7 966
                  3|7 6 9 9 1 119
------------------------------
942 47 7 7 96 0 6|6 1 8 6 847
         1|0 7 5 1 125           ↓¹¹ 1,
          |9 4 2 4 778
         ------------
           1|3 2 6 347           ↓¹² 1,
            |9 4 2 478
         ------------
           3|8 3 869             ↓¹³ 4,
           3|7 6 991
         ------------
              6|878              ↓¹⁵ 7,      ↓¹⁷ 8,
              6|597
              -----
                271              ↓¹⁶ 2,
                181

'0'6'0'0'0'0'4'1'0'3'4'0'0'0'0'0'0'0 ↕ 0,0,1,0,6,|0,0,0,0,0,1,1,4,0,7,2,8,
       3 0 2 0 1 5 1 2 1 0 0 8 6 4         9 9 9 5 0 0 3 3 3 0 8 3 5 3
                             2 0 0 0         5 9 9 9 9 7 0 0 0 0 2 0 0
   ------------------------------      ------------------------------
   6 0 3 0 2 4 1 8 5 2 1 0 2 8 6 4,      1 0 5 9 5 0 0 0 4 4 4 9 2 8 1,
     1 0 5 9 5 0 0 0 4 4 4 9 2 8 1,
   ------------------------------
   '5 9 2 4 2 9 1 8 4 7 6 5 3 5 8 3 ↑₁₇
```

$$\frac{10^{\cdot}}{3^{\cdot}} = \frac{1^{\cdot}}{{\cdot}3} = \frac{3^{\cdot}}{{\cdot}9}$$

$$\begin{array}{rl} \downarrow,(3) = & 10986122886681\,0969, \\ & 10536051565782630, \quad = {\cdot}1 \uparrow_1 \\ \hline & 12039728043259359\,9, \\ & 5924291847653583, \\ \hline \downarrow,(\pi) = & 11447298858494001\,6, \end{array}$$

A result true to the last figure.

∴ Hyp. log π = 1·14472988584940017· true to seventeen places of figures.

1– 23025850929940456\,8,=$\downarrow$,(10)

2– 46051701859880913\,6,=$\downarrow$,$(10)^2$

3– 69077552789821370\,4,=$\downarrow$,$(10)^3$

4– 92103403719761827\,2,=$\downarrow$,$(10)^4$

5–115129254649702284\,0,= &c.

6–138155105579642740\,8,

7–161180956509583197\,6,

8–184206807439523654\,4,

9–207232658369464111\,2,

$$\begin{array}{r} 11447298858494001\,7, \; ({\cdot}4971498726941338 \\ 9210340371976182\,7 \\ \hline 2236958486517819\,0 \\ 2072326583694641\,1 \\ \hline 164631902823177\,9 \\ 161180956509583\,2 \\ \hline 3450946313594\,7 \\ 2302585092994\,0 \\ \hline 1148361220600\,7 \\ 921034037197\,6 \\ \hline 227327183403\,1 \\ 207232658369\,5 \\ \hline 20094525033\,6 \\ 18420680744\,0 \\ \hline 1673844289\,6 \\ 1611809565\,1 \\ \hline 62034724\,5 \\ 46051701\,9 \\ \hline 15983022\,6 \end{array}$$

∴ The common log of π =

·4971498726941338\,5

hich is true to the last figure.

```
159830226
138155105
---------
 21675121
 20723266
---------
   951855
   921034
---------
    30821
    23026
---------
     7795
     6908
---------
      887
      691
---------
      196
      184
---------
       12
```

4. Required the dual and common logarithm of 7.

$$7^2 \times 2 = 98$$

```
98|00|00                                         ↓2 2,
 1|96|98
--------
99 96|98 00|00 0 0|0 0 0 0|0 0
        2|99 90|94 0 0|0 0 0 0|0 0           ↓4 3,
               2|99 9 0|9 4 0 0|0 0
                       |9 9 9 6|9 8
--------------------------------
99 99 97|93 93 9 1|9 3 9 6 9 8
       1|99 99 9 5|8 7 8 7 8 4               ↓6 2,
                  |9 9 9 9 9 8
--------------------------------
99 99 99 93|93 8 8 8 1 8 4|8 0
          5|99 9 9 9 9 6 3|6 3               ↓8 6,
                      1 5|0 0
--------------------------------
99 99 99 99 93 8 8 8 1 6 3 4 3
               ===============
               6,1,1,1,8,3,6,5,7,
```

$\downarrow$ 0,2,0,3,0,2,0,6,0,6,1,1,1,8,3,6,6,

1990066170633616
·29998500099993
199999900000
5999999970

2020270731751945,
$\downarrow$, (2) = 6931471805599453 3,

7133498878·7746478,
$\downarrow$, $(10)^2$ = 46051701859880913 6,

2) 3891820298110626 58,

$7^{\cdot} = \downarrow^{17}$ 1945910149055313 29, = Dual log $\downarrow^{17}$
184·2068074395236 54

1038420746600767 5
921034037197618 3

117386709403149 2
115129254649702 3

2257454753446 9
2072326583694 6

Quotient.
(·8450980400142568 3
the common log of 7, true to the last figure.

Divisor.
2302585092994045 68) 18512816975 23
1842068074395

9213623128
9210340372

3282756
2302585

980171
921034

59137
46052

13085
11513

1572
1382

190
184

6

The dual logarithms of 2; 3; 4; 5; 6; 7; 8; 9; 10; in the seventeenth position may now be arranged in order for future reference; these numbers are easily verified should any mistake be made by the printer; besides, the system may be almost instantly adapted to dual numbers of any number of digits less than seventeen.

$$\overset{17}{\downarrow}, (2) = 69314718055994533,$$

$$\overset{17}{\downarrow}, (3) = 109861228866810969,$$

$$\overset{17}{\downarrow}, (4) = 138629436111989066,$$

$$\overset{17}{\downarrow}, (5) = 160943791243410035,$$

$$\overset{17}{\downarrow}, (6) = 179175946922805502,$$

$$\overset{17}{\downarrow}, (7) = 194591014905531329,$$

$$\overset{17}{\downarrow}, (8) = 207944154167983599,$$

$$\overset{17}{\downarrow}, (9) = 219722457733621938,$$

$$\overset{17}{\downarrow}, (10) = 230258509299404568,$$

Examples.

Ex. 5. *The common logarithm of a number is* 2·5637851810, *find the corresponding number by a direct process.*

To attempt to solve a question like this before dual arithmetic was invented would be perfectly absurd.

2·5637851810 common logarithm.

2302585092,99	46051701860,	2·
$\dagger$	11512925465,	·5
	1381551056,	6
	69077553,	3
	16118096,	7
	1842068,	8
	115129,	5
	2303,	1
	1842,	8
	23,	1
	$\overset{10}{\downarrow}$ 59033335395,	

∴ The dual log of the required number in the tenth position is

$$\overset{10}{\downarrow}\ 59033335395,$$

$$
\begin{array}{rcl}
 & & \overset{10}{\downarrow}\ 59033335\,3\,9\,5, \\
\downarrow, (10)^2 & = & 46051701\,8\,6\,0, \\
\hline
 & & 129816335\,3\,5 \\
\downarrow, (3) & = & 109861228\,8\,7 \\
\hline
 & & 19955106\,4\,8 \\
 & & \overset{10}{\downarrow}\ 19062035\,9\,6, = \downarrow 2, \\
\hline
 & & 893070\,5\,2 \\
 & & 995033\,0\,8, = \overset{2}{\downarrow} 1, \\
\hline
 & & -\ 101962\,5\,6 \\
'1\ \underset{8}{\uparrow} & & +\ 100050\,0\,3, \\
\hline
 & & -\quad 1912\,5\,3 \\
'2\ \underset{5}{\uparrow} & & +\quad 2000\,0\,1, \\
\hline
 & & 8,7,4,8,
\end{array}
$$

'0'0'1'0'2'0'0'0'0'0 ↑ 300 ↓ 2,1,0,0,0,0,0,8,7,4,8, = the dual number corresponding to the common log 2·5637851810.

```
                12|1 0|0
                  |1 2|1
              ─────────────
                12 2|2 1 0|000|00 + = ↓ 2,1,
     '1 ↑           |1 2 2|210|00 −
       8
              ──────────────────────
                12 2 0 8|7 7900|0 +
                        |2 4417|6 −
                               |1
              ──────────────────────
                12 2 0 8 5 34825
                 . . . ↑           7
                            9766 = ↓ 8
                                   8
                             855 = ↓ 7
                                   9
                              49 = ↓ 4
                                   10
                              10 = ↓ 8
              ──────────────────────
                1·2 2 0 8 5 45505
                              300
              ──────────────────────
                3 6 6·2 5 6 3651500 the required number.
```

It may be remarked that 366·25636515 sideral days = 365·25636515 mean solar days.

Dual numbers like the above, having but a single digit in each position, may be exhibited in a more compact form, as ascending digits are sufficiently distinguished from descending by the prefixed and affixed commas. The above may be expressed in the form

300 ↕ 2,1,'1'0'2'0 8,7,4,8, ;

$3\,(10)^2$ ↕ 2,1,'1'0'2'0 8,7,4,8, ;

or, (3) $_2$↕ 2,1,'1'0'2'0 8,7,4,8, .

In the following tabulated form the consecutive bases 1·1 ; 1·01 ; 1·001 ; &c. are represented by dual numbers, none of whose digits exceeds 5, or '5 except the first of each number which is designed to be 10,.

1·0000000001 = $\overset{10}{\downarrow}$ 1, = $\updownarrow$ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,10, 0, 0, 0, 0, 0, 0, 0, 0, 0,

1·000000001 = $\overset{9}{\downarrow}$ 1, = $\updownarrow$ 0, 0, 0, 0, 0, 0, 0, 0, 0,10, 0, 0, 0, 0, 0, 0, 0, 0, '4, '5

1·00000001 = $\overset{8}{\downarrow}$ 1, = $\updownarrow$ 0, 0, 0, 0, 0, 0, 0, 0,10, 0, 0, 0, 0, 0, 0, 0, '4 '5 '0 '0

1·0000001 = $\overset{7}{\downarrow}$ 1, = $\updownarrow$ 0, 0, 0, 0, 0, 0, 0,10, 0, 0, 0, 0, 0, 0, '4 '5 '0 '0 '0 '0

1·000001 = $\overset{6}{\downarrow}$ 1, = $\updownarrow$ 0, 0, 0, 0, 0, 0,10, 0, 0, 0, 0, 0, '4 '5 0, 0, 0, 0, 3, 3,

1·00001 = $\overset{5}{\downarrow}$ 1, = $\updownarrow$ 0, 0, 0, 0, 0,10, 0, 0, 0, 0, '4 '5 '0 '0 '0 3, 3, 0, 0, 0,

1·0001 = $\overset{4}{\downarrow}$ 1, = $\updownarrow$ 0, 0, 0, 0,10, 0, 0, 0, '4 '5 '0 '0 3, 3, 0, 0, '2 '3 0, 5,

1·001 = $\overset{3}{\downarrow}$ 1, = $\updownarrow$ 0, 0, 0,10, 0, 0, '4 '5 '0 3, 3, 0, '2 '3 1, '3 '0 '0 '2 2,

1·01 = $\overset{2}{\downarrow}$ 1, = $\updownarrow$ 0, 0,10, 0, '4 '5 3, 3, '2 '3 3, '5 '1 '5 '1 2, 4, '3 2, 0,

1·1 = $\downarrow$ 1, = $\updownarrow$ 0,10, '4 '2 1, '1 '1 '2 3, '3 '4 '2 1, 5, 0, '3 '5 1, '3 3,

The ease with which the reciprocal of a dual number is found when the number itself is given, will plainly appear by inspecting the subjoined equations; this property is of importance in many inquiries.

If	$x = \downarrow 1,$	then	$\frac{1}{x} = \updownarrow {}'1\ 1,0,1,0,0,0,1,$
	$x^2 = \downarrow 2,$		$\frac{1}{x^2} = \updownarrow {}'2\ 2,0,2,0,0,0,2,$
	$x^3 = \downarrow 3,$		$\frac{1}{x^3} = \updownarrow {}'3\ 3,0,3,0,0,0,3,$
	&c.		&c.
If	$y = \downarrow 0,1,$	then	$\frac{1}{y} = \updownarrow {}'0'1\ 0,1,0,0,0,1,$
	$y^2 = \downarrow 0,2,$		$\frac{1}{y^2} = \updownarrow {}'0'2'0\ 2,0,0,0,2,$
	$y^3 = \downarrow 0,3,$		$\frac{1}{y^3} = \updownarrow {}'0'3'0\ 3,0,0,0,3,$
	&c.		&c.
If	$z = \downarrow 0,0,1,$	then	$\frac{1}{z} = \updownarrow {}'0'0'1'0'0\ 1,0,0,$
	$z^2 = \downarrow 0,0,2,$		$\frac{1}{z^2} = \updownarrow {}'0'0'2'0'0\ 2,0,0,$
	$z^3 = \downarrow 0,0,3,$		$\frac{1}{z^3} = \updownarrow {}'0'0'3'0'0\ 3,0,0,$
	&c.		&c.

Again,

If	$p = {}'1 \uparrow$	then	$\frac{1}{p} = \downarrow\ 1,1,0,1,0,0,0,1,$
	$p^2 = {}'2 \uparrow$		$\frac{1}{p^2} = \downarrow\ 2,2,0,2,0,0,0,2,$
	$p^3 = {}'3 \uparrow$		$\frac{1}{p^3} = \downarrow\ 3,3,0,3,0,0,0,3,$
	&c.		&c.

If q = '0'1 ↑ then $\frac{1}{q}$ = ↓ 0,1,0,1,0,0,0,1,

q^2 = '0'2 ↑ $\frac{1}{q^2}$ = ↓ 0,2,0,2,0,0,0,2,

q^3 = '0'2 ↑ $\frac{1}{q^3}$ = ↓ 0,3,0,3,0,0,0,3,

&c. &c.

If r = '0'0'1 ↑ then $\frac{1}{r}$ = ↓ 0,0,1,0,0,1,0,0,

r^2 = '0'0'2 ↑ $\frac{1}{r^2}$ = ↓ 0,0,2,0,0,2,0,0,

r^3 = '0'0'3 ↑ $\frac{1}{r^3}$ = ↓ 0,0,3,0,0,3,0,0,

&c. &c.

The reciprocal of

↓ 3,2,5,7,4,9,8,3, is ↕'3'03,5,0,5,0,2,

since ↓, 3,2,5,7,4,9,8,3, = 31157853,

and '31157853 = ↕'3'03,5,0,5,0,2,

CHAPTER V.

SOLUTIONS OF IMPORTANT PROBLEMS, DESIGNED AS MODELS AND EXAMPLES OF CONCISE METHODS OF OPERATING, AND SUCCINCT PROCESSES OF INVESTIGATING.

THE recapitulations and short methods of obtaining results instituted in this chapter are required, because previously our objects were more to establish principles by the simplest means, to show the accuracy that might be arrived at, even by clumsy and proscribed operations, and to draw out particular features of Dual Arithmetic in bold relief; rather than to enter upon the generalization of the science, or the facilitation of the art by the introduction of compendious methods of calculation.

RECAPITULATION OF THE GENERAL NOTATION.

Let $$\mathrm{N} = \mathrm{U} \times \mathrm{V} = \mathrm{UV},$$
that is, let the natural number N be equal to the product of the two natural numbers U and V.

Then $$\downarrow,(\mathrm{N}) = \downarrow,(\mathrm{U}) + \downarrow,(\mathrm{V}), \qquad (1),$$
that is, the dual logarithm of N is equal to the dual logarithm of U plus the dual logarithm of V. Observe that the comma ($\downarrow,$) is at the barb of the arrow in (1).

If $$\begin{aligned} \mathrm{U} &= m\!\downarrow\! n\; u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, \\ &= m\!\downarrow\! n\; 0,0,0,0,0,0,0,u, \\ &= m\!\downarrow\! n\; u, \\ \downarrow,(\mathrm{U}) &= m\!\downarrow,\! n\; u_1, u_2, u_3, \ldots\ldots \\ &= m\!\downarrow,\! n\; u, \\ &= u, + m\downarrow,(10) + n\downarrow,(2) \end{aligned}$$

m and n represent powers of 10 and 2; the comma placed at the barb of the arrow indicates that the dual logarithm of the expression is taken, that is, the dual logarithm of

$$10^m \times 2^n \times \downarrow u_1, u_2, u_3, \ldots$$

written, $$ {}_m\!\downarrow_n u_1, u_2, u_3, \ldots$$

is expressed by placing a comma at the barb of the arrow as in (2). $u,$ is the dual logarithm of $\downarrow u_1, u_2, u_3, \ldots.$ and is written

$$\downarrow, u_1, u_2, u_3, \ldots\ldots = u, \qquad (3).$$

$\downarrow u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8,$ being equal to $\downarrow 0,0,0,0,0,0,0,u,$ or $\overset{8}{\downarrow} u,$; and might be written, $u_1, u_2, u_3, n_4, u_5, u_6, u_7, n_8, \downarrow$ or $u, \downarrow$ when the reduction is being reversed.

$$\begin{aligned} N &= V\ {}_m\!\downarrow_n u_1, u_2, u_3, \ldots. \\ &= V\ {}_m\!\downarrow_n u, \end{aligned}$$

Then
$$\downarrow,(N) = \downarrow,(V) + {}_m\!\downarrow,_n u,$$
$$\therefore \quad \downarrow,(N) = \downarrow,(V) + m \downarrow,(10) + n \downarrow,(2) + u,; \qquad (4).$$

$$u, = \downarrow, u_1, u_2, u_3, \ldots.$$

The dual logarithm of N is equal to the dual logarithm of V plus m times the dual logarithm of 10; plus n times the dual logarithm of 2, plus $u,$ the dual logarithm of the dual number $\downarrow u_1, u_2, u_3, \ldots.$ (4).

Let $V = {}'v_1{}'v_2{}'v_3{}'v_4{}'v_5{}'v_6{}'v_7{}'v_8 \uparrow$ and may be written $\uparrow {}'v_1{}'v_2{}'v_3{}'v_4{}'v_5{}'v_6{}'v_7{}'v_8$
$= {}'0'0'0'0'0'0'0'v \uparrow$ and may be written $\uparrow {}'0'0'0'0'0'0'0'v$
$= {}'v \underset{8}{\uparrow} = {}'v \uparrow$ the eight being omitted in all cases. The digits may be on the right or left of $\uparrow$ when not connected with independent ascending dual numbers.

Then

$$\downarrow,(V) = {}'v_1{}'v_2{}'v_3 \ldots. {}'\!\uparrow = {}'v \quad \text{or} \quad {}'\!\uparrow {}'v_1{}'v_2{}'v_3 \ldots. = {}'v; \qquad (5).$$

In (5) the comma is at the barb of the arrow, and designates that the dual logarithm of the descending dual number $'v_1'v_2'v_3 \ldots \uparrow$ is equal to $'v$; and (5) also indicates that the dual logarithm of V is equal to $'v$ which is negative if $u,$ be considered positive, and on the contrary $'v$ is positive when $u,$ is taken as negative.

Since

$$
\begin{array}{c}
\mathrm{N} = \mathrm{VU} \\
\| \\
'v_1'v_2'v_3'v_4'v_5'v_6'v_7'v_8 \;{}^{m}\!\updownarrow^{n}\; u_1,u_2,u_3,u_4,u_5,u_6,u_7,u_8, \\
= \\
'0'0'0'0'0'0'0'v \;{}^{m}\!\updownarrow^{n}\; 0,\ ,0,0,0,0,0,0,u, \\
= \\
'v \;{}^{m}\!\updownarrow^{n}\; u,
\end{array}
$$

These latter expressions are also put under a logarithmic form by simply placing a comma at the barb of the arrow.

Thus

$$
\begin{array}{c}
\downarrow,(\mathrm{N}) = \downarrow,(\mathrm{VU}) = \downarrow,(\mathrm{V}) + \downarrow,(\mathrm{U}). \\
\| \\
'v_1'v_2'v_3 \ldots \;{}^{m}\!\updownarrow,^{n}\; u_1,u_2,u_3, \ldots \\
\| \\
'v \;{}^{m}\!\updownarrow^{n}\; u, \\
= \\
m\downarrow,(10) + n\downarrow,(2) + (u - v), \\
\text{or,} \\
m\downarrow,(10) + n\downarrow,(2) + '(v - u).
\end{array}
$$

Dual numbers composed of ascending and descending dual digits having but one digit in each position, may be placed to the right or left of $\updownarrow$, because ascending digits are sufficiently distinguished from descending by the accompanying commas, and the digits of both branches are ranged in order from left to right.

$$'0'v_2'0'v_4'0'0'0'v_8 \;{}^{m}\!\updownarrow^{n}\; u_1,0,u_3,0,u_5,u_6,u_7,0,$$

may be written $\quad {}^{m}\!\updownarrow^{n}\; u_1,'v_2u_3,'v_4u_5,u_6,u_7,'v_8;$

$$'v_1'v_2'0'0'0'v_6'v_7'v_8 \;{}^{m}\!\updownarrow^{n}\; 0,0,u_3,u_4,u_5,0,0,0,$$

may be written

$$'v_1'v_2u_3,u_4,u_5,'v_6'v_7'v_8 \;{}^{m}\!\updownarrow^{n};$$

if the leading digit be an ascending one, $\updownarrow$ is in most cases placed on the left, but when the leading digit is a descending one, then $\updownarrow$ is generally put on the right of the compound dual number.

$$'0'v_2'0'v_4'v_5'v_6'v_7'v_8 \underset{4}{\uparrow} (10)^m (2)^n \downarrow n_1'0, u_3, 0,0,0,0,0,$$
$$=$$
$$m\updownarrow n \; u_1, 'v_2 u_3, 'v_4, v_5, v_6, v_7, v_8$$

The dual logarithm of

$$'0'v_2'0'0'v_5'v_6'v_7'v_8 . \updownarrow u_1, 0, u_3, u_4,$$

is thus indicated

$$\updownarrow, u_1, 'v_2 u_3, u_4, 'v_5'v_6'v_7'v_8$$

The dual logarithm of

$$'v_1'v_2 \uparrow \times (R) \times \downarrow 0,0, u_3, u_4, u_5, u_6, u_7, u_8,$$

may be put in the form

$$\downarrow, \{'v_1'v_2 u_4, u_5, u_6, u_7, u_8, \updownarrow R\}$$

or written

$$\downarrow, (R) + 'v_1'v_2 u_4, u_5, u_6, u_7, u_8, '\updownarrow$$

As the ordinal arrangements of both ascending and descending digits are from the left to right, the sign $\downarrow$ $\uparrow$ or $\updownarrow$ may be placed to the right or left of ascending, descending, or mixed dual numbers when all the positions are occupied without giving the expression an ambiguous meaning. Yet this change of sign ($\downarrow$ $\uparrow$ $\updownarrow$) from left to right, or *vice versâ*, may be employed to designate the reduction of a natural number to a dual number, and the converse operation.

Let N be a natural number reduced to a dual number

$$m\downarrow n \; u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8,$$

then $\qquad N = m\downarrow n \; u_1, u_2, u_3, \ldots\ldots$

and let $\qquad m\downarrow n \; u_1, u_2, u_3, \ldots\ldots$

be reduced to the natural number N, then

$$u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, \; n\downarrow m = N.$$

Again, if

$$'v_1 'v_2 'v_3 'v_4 'v_5 'v_6 'v_7 'v_8 \uparrow = \mathrm{M}\,; \quad \mathrm{M} = \uparrow 'v_1 'v_2 'v_3 'v_4 'v_5 'v_6 'v_7 'v_8$$

If $\mathrm{P} = \updownarrow u_1, 'v_2 'v_3 u_4, u_5, 'v_6 'v_7 'v_8$

then $u_1, 'v_2 'v_3 u_4, u_5, 'v_6 'v_7 'v_8 \updownarrow = \mathrm{P}\,;$

if $\mathrm{Q} = 'v_1 u_2, 0, 'v_4 u_5, 'v_6 0, 0, \;{}_n\!\updownarrow_m$

also ${}_m\!\updownarrow_u \,, v_1 u_2, 0, 'v_4 u_5, 'v_6 0, 0, = \mathrm{Q}.$

$$\downarrow,(\mathrm{N}) = {}_m\!\downarrow,{}_n \; u_1, u_2, u_3, \ldots\ldots = u,$$

then $u, \downarrow = {}_m\!\downarrow_n \; u_1, u_2, u_3, \ldots\ldots = \mathrm{N}$

or $\overset{8}{\downarrow} u, = \cdot\!\downarrow u_1, u_2, u_3, \ldots\ldots = \mathrm{N},$

hence the law is manifest.

Since $\mathrm{N} = 'v \; {}_m\!\updownarrow_n \; u,$

$$\therefore \quad \frac{\mathrm{N}}{10^m\, 2^n} = 'v \downarrow u,$$

$$\therefore \quad \downarrow,\left(\frac{\mathrm{N}}{10^m\, 2^n}\right) = 'v \downarrow, u, = (u - v), \quad \text{or} \quad = '(v - u)$$

according as v is greater or less than u. See pp. viii. ix. 2, 3, 4, 11.

PROBLEMS.

Ex. 1. *Suppose the dual logarithms of* 10· *and* 2· *to be forgotten, it is required to reproduce them by an easy direct operation.*

Put $x = 2$ and $10 = y$

$$\frac{2^3}{10} = \frac{x^3}{y} = \cdot 8 = '2'1'2'3'7'1'1'7 \uparrow; \text{ which put } = a.$$

$$\frac{2^{10}}{10^3} = \frac{x^{10}}{y^3} = 1\cdot 024 = '0'0'0'1'8'2'1'3 \updownarrow 0,2,4,0,0,0,0,0, ; \text{ which put} = b.$$

Then $\frac{x^3}{y} = a \qquad \therefore \; y = \frac{x^3}{a}$

$\frac{x^{10}}{y^3} = b \qquad \therefore \; y^3 = \frac{x^{10}}{b}$

$$\therefore \; \frac{x^9}{a^3} = \frac{x^{10}}{b} \quad \text{or} \quad \frac{b}{a^3} = x$$

$$\therefore \; \downarrow,(x) = \downarrow,(b) - 3\downarrow,(a) = \downarrow,(2)$$

and $\downarrow,(y) = 3\downarrow,(x) - \downarrow,(a) = \downarrow,(10)$

Calculation.

$\downarrow,(a) = '22314355$ then minus

$$\begin{aligned} \downarrow,(b) &= 2371653, \\ 3\downarrow,(a) &= \underline{66943065,} \\ \downarrow,(2) &= 69314718, \end{aligned}$$

$$\begin{aligned} 3\downarrow,(x) &= 207944154, \\ \text{minus } \downarrow,(a) &= \underline{\;\;22314355,} \\ & \; 230258509, = \downarrow,(10). \end{aligned}$$

Ex. 2. Let it be required to find the dual logarithms of the bases 1·1 ; 1·01 ; 1·001 ; *and* ·9 ; ·99 ; ·999 ; *by simple, direct, and independent operations.*

In the first example a few additions and subtractions will show that $\cdot 8 = '2'1'2'3'7'1'1'7 \uparrow$ and $1\cdot 024 = '1'8'2'1'3 \overset{2}{\underset{4}{\updownarrow}} 2{,}4$, but the reduction of these and other dual numbers to dual logarithms requires the application of the Rules pp. 15 to 22.

For example,

$$\begin{array}{r} '2'1'2'3'7'1'1'7 \uparrow \\ 100510 \\ 72104 \\ 34 \\ \hline '22314355 \end{array}$$

See Rule, page 19.

$$\begin{array}{r} '0'0'0'1'8'2'1'3 \updownarrow 0,2,4,0,0,0,0,0, \\ 001020 \\ \hline 2389800 \\ 66 \\ \hline 2389866 \\ 18213 \\ \hline 2371653, \end{array}$$

See Rules, pages 15, 19.

The Rules by which these simple reductions are made, depend upon the dual logarithms required in the present question.

Because (Introduction, xii.),

$$\cdot 9 \downarrow 1,1,0,1,0,0,0,1, = 1.$$
$$\cdot 99 \downarrow 0,1,0,1,0,0,0,1, = 1.$$
$$\cdot 999 \downarrow 0,0,1,0,0,1,0,0, = 1.$$

$$\therefore \downarrow, (\cdot 9) + \downarrow, 1,1,0,1,0,0,0,1, = 0.$$
$$\downarrow. (\cdot 99) + \downarrow, 0,1,0,1,0,0,0,1, = 0.$$
$$\downarrow, (\cdot 999) + \downarrow, 0,0,1,0,0,1,0,0, = 0.$$

Since the dual log of 1· = 0.

Again, a few simple additions and subtractions will establish the following equations.

$$1\cdot 1 = \downarrow 1,0,0,0,0,0,0,0,$$
$$\|$$
$$'0'0'4'2'0'1'1'2 \updownarrow 0,10,0,0,1,0,0,0,$$
$$1\cdot 01 = \downarrow 0,1,0,0,0,0,0,0,$$
$$\|$$
$$'0'0'0'0'4'5'0'0 \updownarrow 0,0,10,0,0,0,3,3,$$
$$1\cdot 001 = \downarrow 0,0,1,0,0,0,0,0,$$
$$\|$$
$$0'0'0'0'0'0'4'5 \updownarrow 0,0,0,10,0,0,0,0,$$

In the first place we have to find the dual logarithm of

$$(1\cdot 0001)^{10} = \downarrow, 0,0,0,10,0,0,0,0,$$

$$\downarrow, (1\cdot 0001)^{10} = 10 \downarrow, (1\cdot 0001)$$

For ten consecutive digits

$$\begin{array}{r}
1\cdot 0001 = \downarrow 0,0,0,1,0,0,0,0,0,0,\\
\|\\
'0'0'0'0'0'0'0'0'4'5 \updownarrow 0,0,0,0,10,0,0,0,0,0,\\
45\\
\hline
999955,
\end{array}$$

$\therefore$ To eight digits $\downarrow, (1\cdot 0001) = 9999,\cdot 55$ or $10000,$

$\therefore$ $\downarrow, (1\cdot 0001)^{10} = 10 \downarrow, (1\cdot 0001) = 99995,$ nearly.

$$\begin{array}{c}
\therefore \quad \downarrow, (1\cdot 001) = \downarrow, 0,0,1,0,0,0,0,0,\\
\|\\
'0'0'0'0'0'0'4'5 \updownarrow, 0,0,0,10,0,0,0,0,\\
=\\
99995 - 45 = 99950, \qquad \text{(A)}.
\end{array}$$

Then it is evident that

$$\downarrow, (1\cdot 001) = \downarrow, 0,0,1,0,0,0,0,0 \quad = 99950,$$

$$\therefore \quad \downarrow, (1\cdot 001)^{10} = 10 \downarrow, (1\cdot 001) = \downarrow, 0,0,10,0,0,0,0,0, = 999500,$$

$$\begin{array}{c}
\therefore \quad \downarrow, (1\cdot 01) = \downarrow, 0,1,0,0,0,0,0,0,\\
\|\\
'0'0'0'0'4'5'0'0 \updownarrow 0,0,10,0,0,0,3,3,\\
=\\
\begin{array}{r}
999500,\\
33,\\
'4500\\
\hline
995033,
\end{array} \qquad \text{(B)}.
\end{array}$$

$$\downarrow, (\cdot 999) + \downarrow, 0,0,1,0,0,1,0,0, = 0,$$

but $\downarrow, 0,0,1,0,0,1,0,0, = 99950, + 100, = 100050,$ from (A).

$$\therefore \quad \downarrow, (\cdot 999) = -\ 100050, \quad \text{written} \quad {}'100050$$

And because

$$'0'0'1'0'0'0'0'0\,'\!\uparrow\ = \downarrow, (\cdot 999) = {}'100050 \qquad (C).$$

$$\therefore \quad '0'0'4'0'0'0'0'0\,'\!\uparrow\ = {}'400200$$

$$\downarrow, (\cdot 99) + \downarrow, 0,1,0,1,0,0,0,1, = 0$$

but from

$$(B) \quad \downarrow, 0,1,0,1,0,0,0,1, = 995033, + 10001, = 1005034,$$

$$\therefore \quad \downarrow, (\cdot 99) = -\ 1005034, \quad \text{written} \quad {}'1005034 \qquad (D).$$

$$\therefore \quad '0'1'0'0'0'0'0'0\,'\!\uparrow\ = \downarrow, (\cdot 99) = {}'1005034$$

$$\begin{array}{rlll}
 & \downarrow, (1\cdot 1) = \downarrow, 1,0,0,0,0,0,0,0, & & \\
 & \quad \Vert & & \\
0'0'4'2'0'1'1'2 & \updownarrow, 0,10,0,0,1,0,0,0 & & \\
 & \quad \Vert & & \\
\downarrow, 0,10, = & 9950330, & (B) \times 10. & \\
 & 1000, & & \\
\hline
 & 9951330, & & \\
'0\,0'4\,'\!\uparrow\ = & {}'400200 & (C) \times 4. & \\
\hline
 & 9551130, & & \\
 & '20112 & & \\
\hline
 & 9531018, & (E). &
\end{array}$$

$$\therefore \quad \downarrow, (1\cdot 1) = \downarrow, 1,0,0,0,0,0,0,0, = 9531018,$$

$$\downarrow, (\cdot 9) + \downarrow, 1,1,0,1,0,0,0,1, = 0.$$

But from (B) and (E)

$$\downarrow, 1,1,0,1,0,0,0,1 = 9531018, + 995033, + 10001,$$

$$\therefore \quad \downarrow, (\cdot 9) = -10536052, \quad \text{written} \quad '10536052$$

$$\therefore \quad '1'0'0'0'0'0'0'0'\uparrow = \downarrow, (\cdot 9) = '10536052 \qquad \text{(F)}.$$

Ex. 3. *Let it be required to reduce by the shortest processes, and least possible amount of calculation, the seven natural constant numbers (a), (b), (c), (d), (e), (f), (g), to the most simple dual numbers, and to dual logarithms best adopted to logarithmic computation.*

13·59593 (*a*), density of mercury at the temperature of 0° centigrade, according to Regnault.

1·7329625 (*b*), cubic inches of water in an ounce avoirdupois, at the temperature of 62° Fahrenheit.

20853654 (*c*), feet, the semi-axis minor of the earth, according to Bessel.

3437·7468 (*d*), length of the radius in minutes.

39·37079 (*e*), inches in a metre.

61·09908 (*f*), cubic inches in a litre.

9·8087952 (*g*), metres, the accelerating force of gravity.

In reducing natural numbers to dual numbers of the simplest form by the shortest and easiest methods of calculation, any natural numbers, as (*a*), (*b*), (*c*), (*d*), (*e*), (*f*), (*g*), that may be selected, must occupy one or other of the positions between the limiting natural numbers I., II., III., &c., and reduced in a similar manner to (*a*), (*b*), (*c*) .... See "The Young Dual Arithmetician," pp. 83 to 98, and the present work, pp. 7 to 12.

I. 100000000

$(a) \qquad = 13{\cdot}59593 \qquad \therefore\ a = 1\downarrow 3,2,1,3,5,4,7,1,$

II. 141421356

$\dfrac{(b)}{2} \qquad = {\cdot}86648125 \qquad \therefore\ b = {}'1'4\ \overset{3}{\updownarrow}1\ 2,2,4,8,1,1,$

III. 200000000

$\dfrac{(c)}{2} \qquad = 10426827{\cdot} \qquad \therefore\ c = {}'3'4'1\ {}_{7}\overset{2}{\underset{6}{\updownarrow}}1\ 4,2,$

IV. 266000000

$(d) \times 4 \qquad = 13750{\cdot}9872 \qquad \therefore\ d = {}'2'5'4'5'0\ {}_{4}\overset{4}{\underset{4}{\updownarrow}}\bar{2}\ 3,3,3,$

V. 345000000

$(e) \times 2 \qquad = 78{\cdot}74158 \qquad \therefore\ e = {}'2'3'0'1'2'5'7'8\ {}_{1}\overset{3}{\updownarrow}\bar{1}\ 2$

VI. 500000000

$(f) \times 2 \qquad = 122{\cdot}19816 \qquad \therefore\ f = {}'1\ {}_{2}\underset{4}{\updownarrow}\bar{1}\ 2,1,0,0,0,3,1,1$

VII. 707106780

$(g) \qquad = 9{\cdot}8087952 \qquad \therefore\ g = {}'2\ {}_{1}\overset{4}{\underset{2}{\updownarrow}}\ 7,9,5,0,8,$

VIII. 999999999

Dual logarithms best adopted to logarithmic computation.

$$\downarrow,(1\cdot359593) = 30718541,; \qquad \downarrow,(1\cdot7329625) = 54983241,;$$
$$\uparrow,(2\cdot08533654) = 73494409,; \qquad \downarrow,(\cdot34377468) = '106751434;$$
$$\downarrow,(\cdot7874158) = '23899883; \qquad \downarrow,(\cdot6109908) = '49267338;$$
$$\uparrow,(\cdot98087952) = '1930560.$$

See Article II. pp. 12 to 14, respecting the general tables where these logarithms may be found by inspection.

Ex. 4. *Reduce such unwieldy dual numbers as* (A), (B), (C), *to the most convenient forms for reduction to natural numbers. Also find the dual logarithms which determine the corresponding natural numbers through merely inspecting the general tables.*

(A). $'8'5'7'6'9'2'3'4\uparrow \times \frac{10^2}{2^3}$ written $'8'5'7'6'9'2'3'4\ {}_{\bar{3}}\!\uparrow_2$

(B). $\frac{2}{10^4} \times \downarrow 7,8,9,4,5,6,3,4,$ written ${}_{\bar{4}}\!\downarrow_1 7,8,9,4,5,6,3,4,$

(C). $'3'7'7'6'5'5'4'3\uparrow 2^2 10^4 \downarrow 9,3,7,8,4,6,6,7,$
written
$'3'7'7'6'5'5'4'3\ {}_4\!\updownarrow_2\ 9,3,7,8,4,6,6,7,$

Reduction of (A).

$$'8'5'7'6'9'2'3'4'\uparrow = '90083170 = '20768452 - \downarrow,(2)$$

$$\therefore\ '8'5'7'6'9'2'3'4\ {}_{\bar{3}}\!\uparrow_2 = '2\ {}_{\bar{1}}\!\overset{3}{\updownarrow}_2\ 3,0,3,8,0,2, = 5\cdot077789524$$

To find the most convenient log to enter the tables with.

$$'90083170 - 3\downarrow,(2) + 2\downarrow,(10) = 162489695,$$

and

$$162489695, - \downarrow,(10) = '67768814 = \downarrow,(\cdot50778952)$$

found through mere inspection.

Reduction of (B).

$$\downarrow, 7,8,9,4,5,6,3,4, = 6307856, + \downarrow, (2)$$

$$\therefore \quad \bar{4}\downarrow,_1\ 7,8,9,4,5,6,3,4, = \bar{4}\downarrow,_2\ 0,6,3,3,7,8,0,8, = \cdot 000426044192$$

Entering the table with '85321217 the corresponding natural number will be found $= \cdot 426044192$.

$$'85321217 \uparrow^{\bar{8}} = \bar{4}\downarrow_2\ 0,6,3,3,7,8,0,8,$$

Reduction of (C).

$$'3'7'7'6'5'5'4'3\ {}_4\updownarrow_2\ 9,3,7,8,4,6,6,7,$$
$$= '2'0'1'0'0'0'0'0\ {}_4\updownarrow_3\ 0,2,0,0,6,6,6,1, = 66040\cdot 77616$$

The tables must be entered with the dual log '41489782 to which corresponds the natural number ·6604077616; for

$$'3'7'7'6'5'5'4'3 \updownarrow,_2\ 9,3,7,8,4,6,6,7, = '41489782$$
$$= 50139291, + 2\downarrow, (2) - \downarrow, (10)$$

and the decimal point has to be moved five places to the right, which brings ·6604077616 to 66040·77615.

Useful Practical Criteria.

☞ If a student has sufficient skill to solve the preceding four examples without the use of tables or other extraneous aids, it may be fairly presumed that he understands the elements of dual arithmetic.

Ex. 5. *Find a convenient dual number to represent* $\frac{1}{g^{\frac{1}{2}}}$, *when* $g = 32\cdot 1816762$.

$$\downarrow, (g) = 347139724,$$

$\frac{5}{7}$ of 347139724, = 247956946, the reciprocal of which is '247956946 = '17698437 $- \downarrow,(10)$, which may readily be put under the simple dual form

$$'2'0'0'1'1'2'3'3 \; 1\updownarrow \; 0'3'4' = \frac{1}{g^{\frac{1}{4}}}$$

Ex. 6. *Reduce* $\frac{1}{\sqrt{(2536\cdot92172)^2 + (635\cdot297388)^2}}$ *to a simple dual number.*

Put $a = 2536\cdot92172$ and $b = 635\cdot297388$;
then the expression becomes

$$\frac{1}{a\sqrt{1 + \left(\frac{b}{a}\right)^2}};$$

therefore

$$\downarrow,\left[\frac{1}{a\sqrt{1 + \left(\frac{b}{a}\right)^2}}\right] = -\left[\frac{1}{2}\downarrow,\left(1 + \left(\frac{b}{a}\right)^2\right) + \downarrow,(a)\right]$$

$$\downarrow,\left(\frac{b}{a}\right)^2 = \downarrow,\left(\frac{\cdot635297388}{2\cdot53692174}\right)^2 = 2\,('45366213 + '93095145)$$

$$= \downarrow,(\cdot06271045) = '46664207 - \downarrow,(10)$$

$$\therefore \quad \downarrow,\left[1 + \left(\frac{b}{a}\right)^2\right] = \downarrow,(1\cdot06271045) = 6082268,$$

$$\therefore \; -\left[\frac{1}{2}\downarrow,\left\{1 + \left(\frac{b}{a}\right)^2\right\} + \downarrow,(a)\right] = '3041134 + '93095145 - 3\downarrow,(10)$$

$$'3'0'2'0'0'0'0'0 \; 3\updownarrow1 \; 0,5,0,1,1,5,2,9, = '96136279 - 3\downarrow,(10)$$

But $\downarrow,(\cdot38237143) = '96136279$

Hence '3'0'2 $\overset{2}{3\updownarrow1}$ 5,0,1,1,5,2,9, is the simple dual number and ·00038237143 is the natural number of this reciprocal. (21), p. 23.

Ex. 7. *Find the reciprocals of the dual numbers* '3'0'0'0'0'0'0'0 ↓ ;
↓ 0,5,0,4,3,2,1,7, ; *and* '3'0'1'0'0'0'0'0 ↕ 0,5,0,4,3,2,1,5.

The ease with which the reciprocals of dual numbers may be found greatly facilitates the work of calculating the roots of equations.

3'0'0'0'0'0'0'0 '↑ = '·31608156 ;

reciprocal

31608156, = ↓, 3,0,0,3,0,0,0,3,

↓, 0,5,0,4,3,2,1,7, = 5018382, ;

reciprocal

$'5018382 = '5 \underset{2}{\overset{5}{\updownarrow}} 6,7,8,6,$

'3'0'1'0'0'0'0'0 ↕, 0,5,0,4,3,2,1,5,

‖

'26689836

reciprocal

26689836,

‖

'2'5'6'0'0'0'0'0 ↕ 0,0,0,0,7,7,3,8,

See Rules, pp. 15 to 22.

Ex. 8. *Find the roots of the quadratic equation* $x^2 + ax + b = 0$, *and give the results when* a = 2108· *and* b = 3844.

Divide by x, then,

$$x - a + \frac{b}{x} = 0$$

$$\therefore \quad x + \frac{b}{x} = a$$

$y\sqrt{b}$ being substituted for x this last equation becomes

$$y\sqrt{b} + \frac{b}{y\sqrt{b}} = a$$

$$\therefore \quad y + \frac{\text{I}}{y} = \frac{a}{\sqrt{b}}, \text{ which call A.}$$

When $\frac{a}{\sqrt{b}}$ is numerically less than either $+ 2$ or $- 2$ the roots are imaginary, since every number, whole or fractional, positive or negative, added to its reciprocal, gives results which cannot be numerically less than $+ 2$ or $- 2$.

In all such cases y may be put $= {}'v \ldots \uparrow \frac{a}{\sqrt{b}}$ the reciprocal of which is $\frac{\sqrt{b}}{a} \downarrow v, \ldots$ because $\frac{a}{\sqrt{b}}$ must always be greater, and, at the same time, nearly equal to y. Besides, this substitution does not require the application of other particular tests or criteria.

Then putting $r = \frac{a}{\sqrt{b}}$ and $s = \frac{\sqrt{b}}{a}$, and supposing r to be taken greater or less than $\frac{a}{\sqrt{b}}$ to facilitate the calculation, $'v$ and $v,$ vary in accordance, but may be found as follows:—

$${}'v \ldots \uparrow r + s \downarrow v, \ldots = \text{A}$$

$$\therefore \quad r(\text{I} \overset{\uparrow}{+} [v) + s(\text{I} \overset{\downarrow}{+} [v) = \text{A}$$

$$\therefore \quad \overset{\uparrow}{+} r [v + \overset{\downarrow}{+} s [v = \text{A} - (r + s)$$

In order to find a convenient value and position for v, this last expression may be put under the form

$$- rv + sv = \text{A} - (r + s)$$

$$\therefore \quad v = \frac{\text{A} - (r + s)}{-(r - s)},$$

and may be positive or negative when the process is continued.

$$\sqrt{b} = \sqrt{3844^{\cdot}} = 62^{\cdot}; \quad \frac{a}{\sqrt{b}} = \frac{2108}{62} = 34^{\cdot} \qquad \text{(A)},$$

$$\therefore \quad r = 34^{\cdot} \text{ and } s, \text{ reciprocal of } r, = \frac{1}{34} = {\cdot}029411765$$

$$v = \frac{A - (r + s)}{-(r - s)} = \frac{-({\cdot}02941\ \ldots\ldots)}{-(+\ 33{\cdot}97058\ \ldots\ldots)} \text{ gives } '0'0'0'9 \uparrow;$$

then we have $\downarrow 0,0,0,9,0,0,0,9$ reciprocal of $'0'0'0'9\uparrow$.

Again, $\quad '0'0'0'9 \uparrow 34 + \dfrac{1}{34} \downarrow 0,0,0,9,0,0,0,9,$

or, $\quad 33{\cdot}96941224 + {\cdot}029438 25 = 33{\cdot}99885049; \quad (r + s).$

$$\therefore \quad \frac{A - (r + s)}{-(r - s)} = \frac{+\ {\cdot}00114951}{-\ 33{\cdot}93997399} \text{ gives } \downarrow 0,0,0,3,3,8,7,$$

then we have $\quad '0'0'0'0'3'3'8'7 \uparrow$ reciprocal of $\downarrow 0,0,0,0,3,3,8,7,$

$\therefore \quad y = 34$ multiplied by $'0'0'0'9'0'0'0'0 \updownarrow 0,0,0,0,3,3,8,7,$

$$\therefore \quad 34 \times 62 = 2108 \text{ mult. by } '9 \overset{5}{\underset{4}{\updownarrow}} 3,3,8,7, = 2089{\cdot}17446$$

Hence quadratic equations may be solved with great ease and certainty, without completing the square.

$$x \text{ also } = 2108^{\cdot} - 2089{\cdot}17446 = 18{\cdot}92554$$

Ex. 9. *Find the roots of the equation* $x^2 - ax + b = 0$ *and apply the general reasoning to the particular case.*

$$x^2 - 1866{\cdot}58714x + 649{\cdot}539 = 0$$

$$\downarrow, \left(\frac{a}{\sqrt{b}}\right) = \downarrow, (a) - \tfrac{1}{2} \downarrow, (b).$$

Since $y\sqrt{b}$ is put $= x$

$$\therefore \quad \downarrow, (x = \downarrow, (y) + \tfrac{1}{2} \downarrow, (b).$$

$\downarrow,(a)$; $\quad\downarrow,(1\cdot86658714) = 62411173,$

$-\tfrac{1}{2}\downarrow,(b)$; $\quad -\tfrac{1}{2}\downarrow,(\cdot649539) \quad = 21574620,$

$$83985793, = \downarrow,(2\cdot31603784)$$

$$\therefore \quad y + \frac{1}{y} = 2\cdot31603784 = \frac{a}{\sqrt{a}}; \qquad (A).$$

Hence the equation has two real roots since (A) is greater than 2·

Put $r = 2\cdot$ then reciprocal $s = \cdot5$;

$$\therefore \quad v = \frac{A - (r + s)}{-(r - s)} = \frac{-\cdot184\ldots}{-\cdot15}$$

which designates that ’1 ↑ is a convenient value for $'v_1$; then we have ↓ 1,1,0,1,0,0,0,1, the reciprocal of ’1 ↑.

Again, $\quad$ ’1 ↑ 2 + ·5 ↓ 1,1,0,1,0,0,0,1,

or, $\quad 1\cdot800 + \cdot555\ldots = 2\cdot355\ldots; \qquad (r + s).$

$$\therefore \quad \frac{A - (r + s)}{-(r - s)} = \frac{-(\cdot039\ldots)}{-(1\cdot24\ldots)}$$

which designates that ’0’3 ↑ is a convenient value for $'v_2$; then we have

↓ 1,4,0,4,0,0,0,1, reciprocal of ’1’3 ↑.

Then, $\quad$ ’1’3 ↑ 2 + $\dot{\cdot5}$ ↓ 1,4,0,4,0,0,0,1,

or, $\quad 1\cdot74653820 + \cdot57256118 = 2\cdot31909\ldots; \qquad (r + s).$

$$\therefore \quad \frac{A - (r + s)}{-(r - s)} = \frac{-(\cdot00306\ldots)}{-(1\cdot1739\ldots)}$$

which shows that ’0’0’2 ↑ is a convenient value for $'v_3$; then as in the foregoing,

↓ 0,0,2,0,0,2,0,0, is the reciprocal of ’0’0’2 ↑.

Lastly,

$$'0'0'2 \uparrow 1{\cdot}74653820 + {\cdot}57256118 \downarrow 0,0,2,0,0,2,0,0,$$

or,

$$1{\cdot}74304688 + {\cdot}57370800 = 2{\cdot}31675488\,; \qquad (r + s).$$

$$\therefore \quad \frac{A - (r + s)}{-(r - s)} = \frac{({\cdot}00071704)}{(1.16933888)} \text{ gives } '0'0'0'6'1'3'2'1 \uparrow$$

$$\therefore \quad y = '1'3'2'6'1'3'2'1 \uparrow 2{\cdot} \text{ and } \downarrow,(y) = 55502113,$$

But,

$$\downarrow,(x) = \downarrow,(y) + \tfrac{1}{2}\downarrow,(b) = 55502113, + '21574620 = 33927493,$$

$$33927493, = \downarrow,(1{\cdot}40392925)\,;$$

$$\therefore \quad x = 1403{\cdot}92925$$

or, $$x = 1866{\cdot}58714 - 1403{\cdot}92925 = 462{\cdot}65789.$$

The value of this new method of finding the roots of quadratic equations becomes very apparent when the coefficients are large, as in the present example; and when closer limits are taken, the work will be much curtailed.

For instance, if a number a little less than $\frac{a}{\sqrt{b}} - \frac{\sqrt{b}}{a}$ be substituted for y in $y + \frac{1}{y}$ the result approaches $\frac{a}{\sqrt{b}}$. In the foregoing solution, if $2{\cdot}31 - \frac{1}{2{\cdot}31} = 1{\cdot}8$ was put $= r$, then the value of y would be found under the simple form $'0'3'2'6'1'3'5'4 \uparrow (1{\cdot}8)$.

The reason is obvious, since the reciprocal of $\frac{a}{\sqrt{b}} - \frac{\sqrt{b}}{a}$ is

$$\frac{a\sqrt{b}}{a^2 - b} = \frac{\sqrt{b}}{a - \frac{b}{a}}\,; \text{ and } \left(\frac{a}{\sqrt{b}} - \frac{\sqrt{b}}{a}\right) + \frac{\sqrt{b}}{a - \frac{b}{a}} \text{ is nearly } = \frac{a}{\sqrt{b}}$$

but evidently greater than it. Hence $'v \uparrow \left(\frac{a}{\sqrt{b}} - \frac{\sqrt{b}}{a}\right)$ is conveniently put for y.

Ex. 10. *Find the roots of the equation $x^2 + ax - b = 0$, and apply the general formulæ to the particular equation $x^2 + 5x - 81 = 0$.*

When the signs of the roots are changed, the equation becomes

$$x^2 - ax - b = 0,$$

therefore

$$x - \frac{b}{x} = + a.$$

Putting $y\sqrt{b}$ for x, as in the last example, the equation becomes

$$y - \frac{1}{y} = \frac{a}{\sqrt{b}}; \qquad \text{(A).}$$

If y be supposed of the form $r \downarrow u, \ldots$ then we have

$$r \downarrow u, \ldots - {}'u \ldots \uparrow s = \mathrm{A},$$

$$\therefore \quad r\,(1 \mathrel{\downarrow\!\!\!\!-} [u) - s\,(1 \mathrel{\uparrow\!\!\!\!-} [u) = \mathrm{A},$$

$$\therefore \quad \mathrel{\downarrow\!\!\!\!-} r\,[u - \mathrel{\uparrow\!\!\!\!-} 1\,[u = \mathrm{A} - (r - s).$$

In order to find a convenient value and positive for u the last equation may be written

$$+ ru + su = \mathrm{A} - (r - s)$$

$$\therefore \quad u = \frac{\mathrm{A} - (r - s)}{r + s}$$

When A is less than 1, $\dfrac{\sqrt{b}}{a} - \dfrac{a}{\sqrt{b}}$ approaches the value of r, but when A is greater than 1, then $\dfrac{a}{\sqrt{b}} + \dfrac{\sqrt{b}}{a}$ approaches the value of r. In the present case, $\dfrac{a}{\sqrt{b}} = \dfrac{5}{9}$, a proper fraction and $1{\cdot}3$ may be put for r as $\dfrac{9}{5} - \dfrac{5}{9} = 1{\cdot}800 - {\cdot}555 = 1{\cdot}245$

$$s = \frac{1}{1·3} = ·76923077$$

$$\therefore \quad u = \frac{A - (r - s)}{r + s} = \frac{·0247863}{2·06923077} = \downarrow 0,1,2,$$

$\downarrow,0,1,2, = 1194933$, reciprocal $'1194933 = '0'1'2 \updownarrow 0,0,0,1,0,2,0,1,$

$$1·3 \downarrow 0,1,2, + (·76923077)'0'1'2 \updownarrow 0,0,0,1,0,2,0,1,$$

or $$1·31561731 - ·76009374 = ·55553357$$

$$\therefore \quad \frac{A - (r - s)}{r + s} = \frac{·00002198}{2·07572105} = \downarrow 0,0,0,0,1,0,5,6,$$

$$\therefore \quad y = 1·3 \downarrow 0,1,2,0,1,0,5,6, = 1·31564121$$

$$\therefore \quad x = 9(131564121) = 11·84077085$$

Ex. 11. *Find the natural sine and log sine of* 10° *by an independent calculation from knowing the natural sine of* 30° *to be equal* $\frac{1}{2}$.

It is well known that if x be the sine of an arc z to radius 1, when m is odd, then, the sine of mz will be

$$mx - \frac{m(m^2-1)}{2·3}x^3 + \frac{m(m^2-1)(m^2-9)}{2·3·4·5}x^5$$
$$\frac{m(m^2-1)(m^2-9)(m^2-25)}{2·3·4·5·6}x^7 + \&c.$$

In the present example $m = 3$, then $m = 3$, $z =$ an arc of 10° to radius 1, and $x =$ the sine of z.

$$\therefore \quad \tfrac{1}{2} = 3x - 4x^3$$

$$\therefore \quad \frac{1}{x}\left(x^3 + \frac{1}{8}\right) = ·75$$

$\frac{1}{6}\downarrow u$, will be a convenient dual form for the value of x, since

x is a fraction, x^3 may be neglected, then $\frac{1}{8x} = \cdot 75$, or $x = \frac{1}{6}$.
Hence the given equation may be put under the dual form

$$\left(\frac{1}{6}\right)^2 \downarrow 2u, + 'u \uparrow \left(\frac{3}{4}\right) = \cdot 73 ; \qquad (A)$$

and the value of u may be found to any required degree of accuracy.

Put $$\left(\frac{1}{6}\right)^2 = r \quad \text{and} \quad \left(\frac{3}{4}\right) = s.$$

$$r(1 \downarrow [2u) + s(1 \uparrow [u) = \cdot 75 ; \qquad (A)$$

$$\downarrow r [2u \uparrow s [u = A - (r + s).$$

In order to find a convenient value of u, this last expression may be put under the form

$$+ 2ru - su = A - (r + s)$$

$$\therefore \quad u = \frac{A - (r - s)}{2r - s} = \frac{-\frac{1}{36}}{-\frac{50}{72}} = \downarrow 0{,}4,$$

$\downarrow, 0{,}4, = 3980132$, reciprocal $= '3980132 = '0'4 \uparrow 0{,}0{,}0{,}4{,}0{,}0{,}0{,}4,$

Again, putting $\left(\frac{1}{6}\right) \downarrow 0{,}4{,}u,$ for x the given equation becomes

$$\left(\frac{1}{6}\right)^2 \downarrow 0{,}8{,}2u, + \left(\frac{3}{4}\right) '0'4'u \uparrow 0{,}0{,}0{,}4{,}0{,}0{,}0{,}4, = \cdot 75 ; \qquad (A).$$

If $$\left(\frac{1}{6}\right)^2 \downarrow 0{,}8, = \cdot 03007935 \text{ be put for } r,$$

and $$\left(\frac{3}{4}\right) '4 \,{}_2\!\!\uparrow^4 4{,}0{,}0{,}0{,}4, = \cdot 72073527 \text{ for } s,$$

then $$u = \frac{A - (r + s)}{2r - s} = \frac{-\cdot 00081462}{-\cdot 66057757} = \downarrow 0{,}0{,}1{,}2{,}3,$$

Because $\downarrow, 0,4,1,2,3,0,0,0,0,0,0, = 41030813597,$
and the reciprocal $'41030813597 = '0'4'0'8'2'9'4'3'0'0'7'5 \downarrow$

Then putting $\left(\frac{1}{6}\right) \downarrow 0,4,1,2,3,u,$ for x, we have

$$r = \left(\frac{1}{6}\right)^2 \downarrow 0,8,2,4,6, = \cdot 030153408486$$

$$s = \left(\frac{3}{4}\right) '0'4'0'8'2'9'4'3'0'0'7'5 \uparrow = \cdot 719849665962$$

$$\therefore \quad u = \frac{A - (r + s)}{2r - s} = \frac{\cdot 000003074448}{\cdot 659542848992} = '4'6'6'1'4'8'4 \underset{6}{\uparrow}$$

$$\therefore \quad x = \left(\frac{1}{6}\right) \downarrow, 0,4,1,2,3,4,6,6,1,4,8,4, = \cdot 17364817766$$

$$\therefore \quad \downarrow, (x) = 551861098745, - \downarrow, (10).$$

Therefore the natural sine of $10° = \cdot 17364817766$ and the dual log sine $= 551861098745, - \downarrow, (10)$.

Ex. 12. *When* s *represents the sine of the arc* a *to radius* 1, *then* $7s - 56s^3 + 112s^5 - 64s^7 =$ *sine of* $(7a)$; *if* $7a = 180°$, $7a = 360°$, $7a = 540°$, *&c., the equation becomes*

$$(2^2s^2)^3 - 7(2^2s^2)^2 + 14(2^2s^2) - 7 = 0;$$

find the three values of $(2s)$.

The given equation may be put under the form

$\frac{(2^2s^2)^{\frac{3}{2}}}{\sqrt{7}} = 2^2s^2 - 1$, which when divided by 2^2s^2 becomes $\frac{2s}{\sqrt{7}} = 1 - \frac{1}{2^2s^2}$, putting $v = 2s$ the last equation becomes

$$\frac{v}{\sqrt{7}} + \frac{1}{v^2} = 1; \qquad \text{(K)}.$$

$$\frac{1}{\sqrt{7}} = \frac{\sqrt{7}}{7} = \cdot 37796447 = a.$$

A value of v being something less than $2\cdot$ put $'u \uparrow (2)$ for v, the equation (K) becomes

$$2a\,(1 \overset{\uparrow}{+} [u) + \tfrac{1}{4}\,(1 \underset{\downarrow}{+} [2u) = 1.$$

$$2a \overset{\uparrow}{+} 2a\,[u + \tfrac{1}{4} \underset{\downarrow}{+} \tfrac{1}{4}\,[2u = 1.$$

$$\therefore \quad \overset{\uparrow}{+}\, 2a\,[u \underset{\downarrow}{+} \tfrac{1}{4}\,[2u = 1 - (2a + \tfrac{1}{4}).$$

or,

$$(-\,2a + \tfrac{1}{2})u = 1 - (2a + \tfrac{1}{4})$$

$$\therefore \quad u = \frac{1 - (2a + \frac{1}{4})}{-\,2a + \frac{1}{2}} = \frac{-\cdot 00592894}{-\cdot 25592894} = {}'0'2 \uparrow.$$

The process being continued, one value of v is found to be

$= (2) \updownarrow {}'0'2'5'3'\ 1,1,5,5,$ usually written $'0'2'5'3 \uparrow (2) \downarrow 0,0,0,0,1,1,5,5,$

$$'0'2'5'3 \updownarrow 0,0,0,0,1,1,5,5, = \cdot 97492802 = \text{sine of } \frac{540^\circ}{7}.$$

Chord or $\qquad 2s = 1\cdot 949856 04.$

Also v may be found

$$= \left(\frac{3}{2}\right) \downarrow, 0,4,1,7,6,2,8,4, = 1\cdot 56365975 = 2s$$

$$s = \cdot 78182988 = \text{sine of } \frac{360^\circ}{7}.$$

The given equation may be put under another form, since

$$\frac{(2^2 s^2)^3}{7} = (2^2 s^2 - 1)^2 \quad \text{or} \quad = (1 - 2^2 s^2)^2$$

$$\therefore \quad \frac{(v^2)^{\frac{3}{2}}}{\sqrt{7}} = 1 - v^2$$

$$\therefore \quad \frac{v^3}{\sqrt{7}} = 1 - v^2$$

$$\therefore \quad \frac{1}{v^2} - \frac{v}{\sqrt{7}} = 1\,;$$

or, $$\frac{1}{v^2} - av = 1\ ; \qquad (K)\ ;$$

$$(a) \text{ being put for } \frac{1}{\sqrt{7}} = \frac{\sqrt{7}}{7} = \cdot 37796447301$$

If v be put $= {}'u\uparrow$ then (K) becomes

$$\downarrow 2u, - {}'u\uparrow (a) = 1$$

$$\therefore \quad 1 \overset{\downarrow}{+} [2u - a(1 \overset{\uparrow}{+} [u) = 1$$

$$\therefore \quad \overset{\downarrow}{+} [2u - \overset{\uparrow}{+} a[u = 1 - 1 + a = a$$

To find a convenient value for u this last equation may be written

$$+ 2u + au = (2 + a)u = a$$

$$\therefore \quad u = \frac{a}{2 + a} = \frac{\cdot 377 \cdots}{2\cdot 377 \ldots} = {}'1\uparrow$$

If $'u\uparrow = {}'1\uparrow$ the square of the reciprocal will be $\downarrow 2,2,0,2,0,0,0,2,$ now equation (K) becomes

$$(1\cdot 23456789) \downarrow 2u_1 - {}'u_1\uparrow (\cdot 34016802) = 1.$$

or, $$b \downarrow 2u_1 - {}'u_1\uparrow (c) = 1$$

$$\therefore \quad b(1 \overset{\downarrow}{+} [2u_1) - c(1 \overset{\uparrow}{+} [u_1) = 1$$

$$\therefore \quad \overset{\downarrow}{+} b[2u_1 - \overset{\uparrow}{+} c[u_1 = 1 - b + c$$

This last expression may be put under the form

$$(+ 2b + c)u_1 = 1 - b + c$$

to find a convenient value for u_1

$$u_1 = \frac{1 - b + c}{2b + c} = \frac{\cdot 10560013}{2\cdot 80930380}$$

which indicates that $'0'3\uparrow$ is a convenient value for $'u_1$.

$\downarrow 0,6,0,6,0,0,0,6,$ is the square of the reciprocal of $'0'3\uparrow$.

By continuing the process the first six digits will be found to be $'1'3'6'3'1'7 \ldots\ldots \uparrow$ The next step not only verifies the foregoing work, but also determines the next five or six digits following.

$$'1'3'6'3'1'7 \ldots\ldots \uparrow$$

reduced to the twelfth position

$$= '141831540281$$

the logarithm of the reciprocal squared will be

$$283663080562, = '0'0'2'2'6'6'4'4'7'8'7'7 \updownarrow 3,$$

$$'1'3'6'3'1'7 \uparrow (\cdot 37796447301) = \cdot 327985258 62\ (c_1)$$

$$'0'0'2'2'6'6'4'4'7'8'7'7 \updownarrow 3, = 1\cdot 32798533128\ (b_1)$$

$$(b_1) \downarrow 2u, - 'u \uparrow (c_1) = 1.$$

$$\therefore \quad b_1(1 \downarrow [2u) - c_1(1 \uparrow [u) = 1$$

$$\downarrow b_1[2u - \uparrow c_1[u = 1 + c_1 - b_1$$

and ultimately, $\quad (+\, 2b_1 + c_1)u = 1 + c_1 - b_1$

$$\therefore \quad u = \frac{1 + c_1 - b_1}{2b_1 + c_1} = \frac{-\ \cdot 00000007266}{+\ 2\cdot 984\ldots\ldots\ldots} = \overset{8}{\downarrow} 2,4,3,4,$$

$$\therefore \quad v = '1'3'6'3'1'7'0'0'0'0'0 \updownarrow 0,0,0,0,0,0,0,2,4,3,4,$$

$$\therefore \quad \frac{v}{2} = \cdot 433883724578 = s = \text{sine of } \frac{180^\circ}{7}.$$

It may be shown by plane geometry, that sine $(3a) = 3s - 4s^3$; sine $(5a) = 5s - 20s^3 + 16s$; sine $(7a) = 7s = 56s^3 + 112s^5 - 64s^7$; &c. (See Leslie's Elements of Geometry. *Prop.* III. p. 356, 1811.) Putting z for $v^2 = 2^2s^2$ and sine $(7a) = 0$, as before, the equation becomes

$$z^3 - 7z^2 + 14z - 7 = 0$$

To take away the second term, substitute $x + \frac{7}{3}$ for z, then the equation becomes

$$x^3 - \frac{7}{3}x + \frac{7}{27} = 0. \qquad \text{(L)}.$$

Let AGFH be a circle, radius $OF = OH = a$, ABC an inscribed equilateral triangle; take any point E, between C and B, make the arc $EC = CF = FG$, and draw the other lines of the figure.

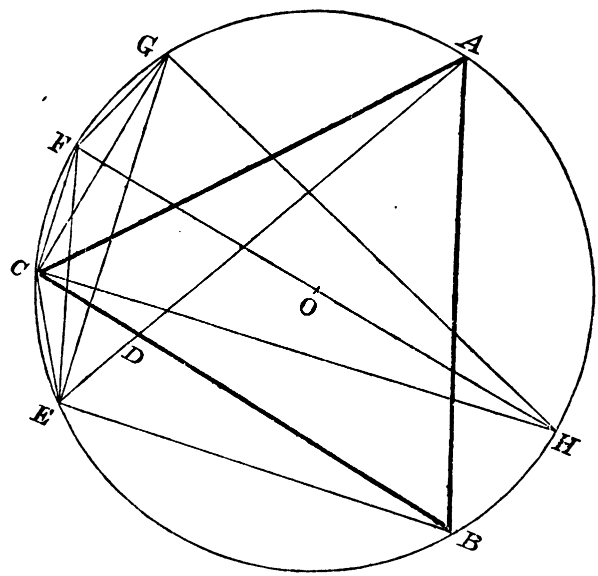


Fig. 2.

Now, by *Prop.* D, Simson's Euclid, VI.

$$AE \times CB = CA \times EB + EB \times AB$$

dividing by $\quad CB = CA = AB$

gives $\quad AE = EB + EC.$

Consequently, if EC and EB represent two of the roots of a cubic equation, wanting its second term, AE with a contrary sign will represent the third root. (Saunderson's Algebra.) Suppose the chord EG of the three arcs EC, CF, FG, to be known and put $= 2b$. It is easily shown that EB is $\frac{1}{3}$ of the arc EBAG.

Let $\qquad x =$ the chord $EC = CF = FG$,

and $\qquad y =$ the chord $EF = CG$,

$$GH = \sqrt{4a^2 - x^2}.$$

The two quadrilateral figures CFGH and EGFC furnish the following equations :—

$$2a \times y = x\sqrt{4a^2 - x^2} + x\sqrt{4a^2 - x^2}$$

$$y \times y = x \times x + 2b \times x$$

$$\therefore \quad x^3 - 3a^2x + 2a^2b = 0 \qquad \text{(M)}.$$

$\therefore$ the chord $CE = x$, will be one of the positive roots of this equation ; and the chord EB of the third part of the arc on the other side of EG must be the affirmative root ; for if x be put for the chord of one-third of the arc EBHG, the equation will be the same. Comparing equations (L) and (M),

$$3a^2 = \frac{7}{3} \qquad \therefore \quad a = \frac{\sqrt{7}}{3} \qquad \text{and} \quad 2a^2 = \frac{14}{9};$$

also, $$2a^2b = \frac{7}{27} \qquad \therefore \quad b = \frac{1}{6} = \text{half EG}.$$

We subjoin the solution of an ingenious geometrical question connected with a class of cubic equations ; it was proposed in "The Lady's, Farmer's, and Mathematical Almanack," Dublin, 1861 ; and reproposed in the same work, 1862 and 1863, by Mr. Matthew Collins. No solution of this question has been before published as far as the Author of the present work is cognizant.

Question.

Prove *geometrically* that the division of a right angle, or the circumference of a circle, into 7 equal parts, can be effected by means of the trisection of another given angle whose tangent is $3\sqrt{3}$.

The following construction may be employed to divide the circumference of a given circle, BTAZ, into seven equal parts by plane geometry, when the arc DP or the arc PnQD of the circle DQnP is trisected.

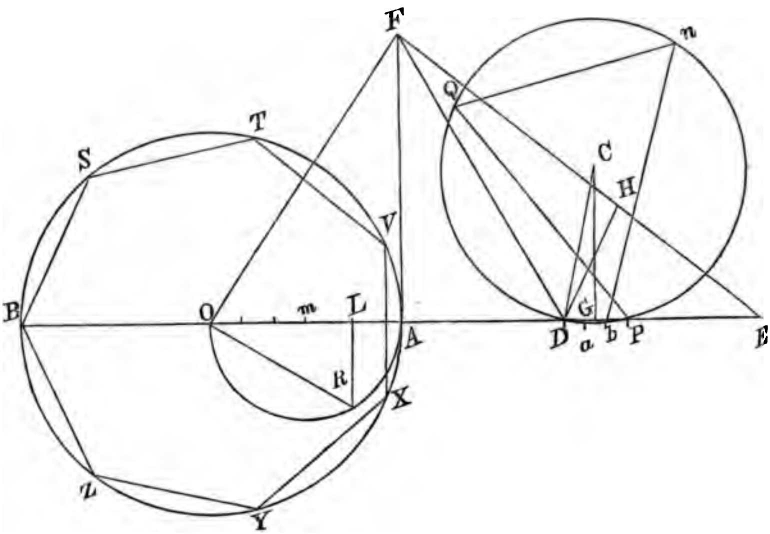


Fig. 3.

Take OA = AD = DE = 1 ; on OD describe the equilateral triangle OFD and draw FE. Draw DH parallel to OF and EH will be the radius of the circle abnQ. Take DP $= \frac{1}{3}$ of DE, or of the radius OA, then the straight line DG $= \frac{1}{2}$ DP $= \frac{1}{6}$ of OA, CG $= \frac{1}{2}$AF.

In the right-angled triangle CDG, CD = HE $= \frac{\sqrt{7}}{3}$ = the radius of the circle abnQ ; DG = GP $= \frac{1}{6}$ of AO ; and CG $= \frac{1}{2}$AF $= \frac{\sqrt{3}}{2}$.

The tangent of the angle $CDG = \frac{CG}{DG} = \frac{\sqrt{3}}{2} \div \frac{1}{6} = 3\sqrt{3}$. Let P$b$ be equal one-third of the arc PbaD, then if bn be the side of an equilateral triangle inscribed in the circle, Pn will be one-third of the greater arc PnQD; Qn, nb, are the sides of an inscribed equilateral triangle. Make PL = PQ, and draw LR perpendicular to OA meeting the semicircle ORA in R, then OR equal to the side of the regular Heptagon BSTVXYZ.

It is evident, from what has gone before, that PQ is the negative value of x in the equation

$$x^3 - \frac{7}{3}x + \frac{7}{27} = 0.$$

$$OP = \frac{7}{3} = 2\tfrac{1}{3} = OA + AD + DP$$

$$OL = -x + \frac{7}{3} = z = 2^2 s^2.$$

But $\qquad AO \times OL = OR^2 \quad \text{or} \quad 1 \times OL = OR^2$

$$\therefore \quad OR = 2s \text{ when } s = \text{sine of } \frac{\pi}{7}.$$

The positive values of x apply to $\frac{2\pi}{7}$ and $\frac{3\pi}{7}$.

Ex. 13. *Find the first twelve and the number of figures in the continued product* 1.2.3.4.5.6 . . . . 18.

It is shown in the Author's work, "The Calculus of Form, a New Science," that

$$\downarrow, (1) + \downarrow, (2) + \downarrow, (3) + \downarrow, (4) + \ldots\ldots \downarrow, (x)$$
$$= \frac{1}{2}\downarrow, (2\pi) + \left(x + \frac{1}{2}\right)\downarrow, (x) - 10^{13}Q;$$

$$Q = x - \frac{B_1}{1\cdot 2}\frac{1}{x} + \frac{B_3}{3\cdot 4}\frac{1}{x^3} - \frac{B_5}{5\cdot 6}\frac{1}{x^5} + \frac{B_7}{7\cdot 8}\frac{1}{x^7} - \ldots\ldots$$

$B_1, B_3, B_5, \ldots$ are the well-known numbers Bernoulli, the first ten being $B_1 = \frac{1}{6}$; $B_3 = \frac{1}{30}$; $B_5 = \frac{1}{42}$; $B_7 = \frac{1}{30}$; $B_9 = \frac{1}{66}$; $B_{11} = \frac{691}{2730}$; $B_{13} = \frac{7}{6}$; $B_{15} = \frac{3617}{510}$; $B_{17} = \frac{43867}{798}$; $B_{19} = \frac{1222277}{2310}$.

$$\begin{array}{rll}
x = & 18{\cdot}0000000000000\ + & \\
\frac{B_1}{1{\cdot}2}\frac{1}{x} = & {\cdot}0046296296296\ - = & \frac{1}{12} \times \frac{1}{18} \\
\frac{B_3}{3{\cdot}4}\frac{1}{x^3} = & {\cdot}0000004762990\ + = & \frac{1}{360} \times \frac{1}{18^3} \\
\frac{B_5}{5{\cdot}6}\frac{1}{x^5} = & {\cdot}0000000004200\ - = & \frac{1}{1260} \times \frac{1}{18^5} \\
\frac{B_7}{7{\cdot}8}\frac{1}{x^7} = & {\cdot}0000000000009\ + = & \frac{1}{1680} \times \frac{1}{18^7} \\
\hline
& 17\ 9953708462503, & = 10^{13}Q
\end{array}$$

$\downarrow,(18) = 28903717578961647 1$, in the 17th position.

$$\begin{array}{rl}
18\tfrac{1}{2}\downarrow,(18) = & 53471877521079047 13, \\
\tfrac{1}{2}\downarrow,(2\pi) = & 9189385332046727 5, \\
\hline
& 5439081605428371988, \text{ in the 17th position.} \\
& 179953708462503, \quad \text{in the 13th position.} \\
\hline
& 363954452080334, = (10^{15})\downarrow 4,8,9,5,3,5,6,0,8,1,1,3,2,
\end{array}$$

$4(10^{15})\downarrow 4,8,9,5,3,5,6,0,8,1,1,3,2, = 6402373705730000$

This result shows that the product consists of 16 places of figures, the first twelve being 640237370573, which is true to the last figure, for by common multiplication

$$1.2.3.4.5 \ldots\ldots 18 = 6402373705728000.$$

The calculation becomes more easy the greater x becomes, as the two following examples will show. This problem of so

much importance in the theory of chances, the calculus of finite differences, definite integrals, &c. received great attention from Euler, Legendre, and Laplace, and yet, with extensive tables and tiresome approximations, very few of the first figures of their results could be depended upon. The dual calculus, in this as in numerous other instances, in a direct manner, removes with ease all impediments.

Ex. 14. *Find the first eight and the number of figures in the continued product* 1.2.3.4.5.6 . . . . 365.

In this example $x = 365\cdot$ and

$$\frac{B_3}{3\cdot4}\frac{1}{x^3} = \frac{1}{360}\cdot\frac{1}{365^3} = \frac{1}{360} \times \frac{1}{48627125},$$

which will not amount to a unit in the eight decimal place;

$$\therefore \quad \text{Q in this case} = x - \frac{B_1}{1\cdot2}\frac{1}{x} = (365\cdot) - (\cdot000\dot{2}2832)$$

$$\therefore \quad (10^8)\text{Q} = 36499977168, \text{ ar. co. } \bar{1}63500022832,$$

$$\downarrow, (3\cdot65) = \quad 12947271676, \text{ in the 10th position.}$$

$$(365\tfrac{1}{2})\downarrow, (3\cdot65) = 4732227797578, \text{ in the 10th position.}$$
$$= 47322277976, \quad \text{in the 8th position.}$$

$$\downarrow, (365) = \downarrow, (3\cdot65) + \downarrow, (10^2)$$
$$(365\tfrac{1}{2})\downarrow, (365) = (365\tfrac{1}{2})\downarrow, (3\cdot65) + \downarrow, (10^{731})$$

These simple preliminary arrangements and calculations being made, the required result is almost instantly obtained, and that, too, without the use of tables or other outlandish dodges.

$$\begin{array}{ll} (365\tfrac{1}{2})\downarrow, (3\cdot65) = & 47322277976, \\ (10^8)\,\text{Q} & \bar{1}63500022832, \text{ ar. co.} \\ \tfrac{1}{2}\downarrow, (2\pi) & 91893853, \\ \hline \end{array}$$

$$\downarrow, (10^{47}) + 92044726, = \; 10914194661, = \downarrow, (10^{47}) + 2\downarrow, 2,3,6,8,3,1,7,3,$$

$$2\downarrow 2,3,6,8,3,1,7,3, = 2\cdot51041286$$

$\therefore$ The first nine figures of the continued product

$$1.2.3.4.5 \ldots\ldots 365. \text{ will be } 251041286$$

and the complete product will consist of 779 places of figures, for $10^{731} \times 10^{47} = 10^{778}$ and $2{\cdot}510\ldots$ multiplied by 10^{778} gives 779 places of figures.

Ex. 15. *Find the first seventeen figures and the number of figures of the continued product* 1.2.3.4.5.6.7. . . . . 1875.

$$\downarrow,(1{\cdot}875) = 62860865942237405, \text{ in the 17th position.}$$

$$(1875\tfrac{1}{2})\downarrow,(1{\cdot}875) = 117895554074666253078, \text{ in the 17th position.}$$

$$\tfrac{1}{2}\downarrow,(2\pi) = 918938533204672 75, \text{ in the 17th position.}$$

$$\begin{array}{l} 117987447927986720353, = \downarrow,(10^{512}) \\ +\ 95091166691581537 \end{array}$$

$$\downarrow,(1875) = \downarrow,(10^{3}) + \downarrow,(1{\cdot}875)$$

$$\therefore\quad (1875\tfrac{1}{2})\downarrow,(1875) = \downarrow,(10^{5626\frac{1}{2}}) + (1875\tfrac{1}{2})\downarrow,(1{\cdot}875);$$

$$(1875\tfrac{1}{2})\downarrow,(1{\cdot}875) = \downarrow,(10^{512}) + 9509116691581537,$$

$$\therefore\quad \tfrac{1}{2}\downarrow,(2\pi) + (1875\tfrac{1}{2})\downarrow,(1875) = \downarrow,(10^{6138\frac{1}{2}}) + 95091166691581537.$$

$$1875^2 = 3515625$$

$$\frac{1}{1875} = {\cdot}00053333333333333 = A$$

$$\frac{A}{(1875)^2} = \frac{1}{(1875)^3} = {\cdot}00000000015170370 = B$$

$$\frac{B}{(1875)^2} = \frac{1}{(1875)^5} = {\cdot}00000000000000004 = C$$

(C) and the succeeding terms may be rejected.

In this example

$$Q = x - \frac{B_1}{1\cdot 2}\frac{1}{x} + \frac{B_3}{3\cdot 4}\frac{1}{x^3}$$

$$= x - \frac{1}{12}\frac{1}{x} + \frac{1}{360}\frac{1}{x^3}$$

$$= (1875\cdot) - (\cdot 000044444444444444)$$

$$+ \cdot 0000000000000042140$$

$$\therefore \quad 10^{17}Q = 187499995555555597696,$$

$$= \downarrow,(10^{814}) + 69568985840279344,$$

$$\therefore \quad \tfrac{1}{2}\downarrow,(2\pi) + (x + \tfrac{1}{2})\downarrow,(x) - 10^{17}Q$$

$$= \downarrow,(10^{5324\frac{1}{2}}) + 25522180851302193,$$

$$= \downarrow,(10^{5324}) \;\; + 140651435501004477,$$

$$\overset{17}{\downarrow} \; 140651435501004477,$$

$$= 4 \downarrow 0,2,0,3,1,9,3,4,7,2,3,7,3,3,2,8,9,$$

$$= 4\cdot 08170321111040128$$

This product will consist of 5325 places of figures, the first seventeen of which will be

408170321111040 13 . . . .

Ex. 16. *Required the first eight figures of, and the number of figures in, the continued product* 366.367.368.369. . . . . 1875.

$\downarrow,(1.2.3 \ldots\ldots\ 1875.) = \downarrow,(10^{5324}) + 140651436,$ (*Ex.* 15.)
$\downarrow,(1.2.3 \ldots\ldots\ 365.) = \downarrow,(10^{778}) + 92044724,$ (*Ex.* 14.)

$$\downarrow,(10^{4546}) + 48606712,$$

But $48606712, = \downarrow,5,0,9,5,2,0,7,2 = \downarrow,(1\cdot 6259091)$.

Consequently, the first eight figures of the continued product 366.367.368.369 . . . . . . 1875, will be 16259091 and the complete product will consist of 4547 figures.

Ex. 17. *Suppose* (*a*) 371, *the number of chances for the happening of an event in a single trial, and* 3597 (*b*), *the number of chances for its failing; find how many trials must be made to have an even chance that the event will happen once.*

Let x be the number of trials.

Then, according to De Moivre

$$\frac{b^x}{(a+b)^x} = \frac{1}{2}$$

$$\therefore \quad 2 = \left\{\frac{a+b}{b}\right\}^x = \left\{1 + \frac{a}{b}\right\}^x$$

$$\therefore \quad x = \frac{\downarrow,(2)}{\downarrow,\left(1+\frac{a}{b}\right)} = \frac{\downarrow,(2)}{\downarrow,(1\cdot10314151)} = \frac{69314718,}{9816207,}$$

$$\therefore \quad x = 7\cdot0612 \text{ or } 7 \text{ times nearly.}$$

Ex. 18. *Suppose* (*a*) *the number of chances for the happening of an event in a single trial, and* (*b*) *the number of chances for its failing; find how many trials* (*r*) *must be made to have an even chance that the event will happen* (*r*) *times at least.*

First put $a = 5$; $b = 1000$; and $r = 3$;

then $\quad \frac{b}{a} = 200 = q; \quad \frac{a}{b} = \cdot005; \quad \frac{a^2}{b^2} = \cdot000025.$

The chance that the event will happen at least three times in x trials is equal to the first $x - 2$ terms of the expansion of

$$\left(\frac{a}{a+b} + \frac{b}{a+b}\right)^x,$$

and this chance by hypothesis is $\frac{1}{2}$. Hence the last three terms of the expansion will be equal to $\frac{1}{2}$, that is

$$b^x + xab^{x-1} + \frac{x(x-1)}{1\cdot2}a^2b^{x-2} = \frac{1}{2}(a+b)^x$$

Dividing by b^x this equation may be written

$$x\left\{\frac{1}{x}+\frac{a}{b}+\frac{a^2}{2b^2}(x-1)\right\}=\frac{\left(1+\frac{a}{b}\right)^x}{\lfloor 2}\,; \qquad (1).$$

Since $a = 5\cdot$ and $b = 1000\cdot$ (1) becomes

$$x\left\{\frac{1}{x}+(\cdot 0000125)x+\cdot 0049875\right\}=\frac{(1\cdot 005)^x}{2}\,; \qquad (2).$$

It is easily shown that x has some value between $100\cdot$ and $1000\cdot$. Put $500\downarrow u,$ for x, then (2) assumes the form

$$100\left\{\frac{1}{500\downarrow u,}+(\cdot 0000125)(500\downarrow u,)+\cdot 0049875\right\}=\frac{(1\cdot 005)^{500\downarrow u,}}{(2)\,(5\downarrow u,)}\,; \quad (3).$$

We may here premise that

$\updownarrow$'1 1,0,1,0,0,0,1,	is the reciprocal of	$\downarrow$ 1,
$\updownarrow$'2 2,0,2,0,0,0,2,	" "	$\downarrow$ 2,
$\updownarrow$'3 3,0,3,0,0,0,3,	" "	$\downarrow$ 3,
&c.		&c.

and *vice versâ.* The same may be said of

$\downarrow$ 0,1, and $\updownarrow$'0'1 0,1,0,0,0,1,; of $\downarrow$ 0,0,1, and $\updownarrow$'0'0'1001,0,0, &c.

$$\downarrow,(1\cdot 005) = 498755,$$

$$\therefore\ 498755, \times 500 = 249377500, = \downarrow,(\cdot 005)^{500}$$

(3) put under a logarithmic form becomes

$$\downarrow,\left\{\frac{1}{5\downarrow u,}+\cdot 625\downarrow u,+\cdot 49875\right\}$$
$$= 249377500, \mathrel{\downarrow\!\!\!\!-}(249377500,)\,[u, - \downarrow,(10) - \downarrow,u,$$
$$= \ 19118991, \mathrel{\downarrow\!\!\!\!-}(249377500,)\,[u, - \downarrow,u,$$

Although the reductions here instituted are extremely simple, yet we have been careful to record every step in establishing a convenient form (4) to operate with, because this is the first equation of the sort ever solved by a direct process.

$$\downarrow,\left\{\frac{\cdot 20000000}{\downarrow u,} + \cdot 62500000 \downarrow u, + \cdot 49875000\right\}$$
$$= 19118991, \downarrow(249377500)\,[u - \downarrow, u, \qquad (4).$$

$$\begin{array}{r} \cdot 20000000 \\ \cdot 62500000 \\ \cdot 49875000 \\ \hline \end{array}$$
$$\downarrow,(1\cdot 32375000) = 28046962,$$

Now let us suppose $u,$ to be in the first position, then for every unit in it the right-hand member of (4) will be increased by 24937750, and diminished by 9531018, hence the increase for $u,$ units will be at least $= (24937750 - 9531018)u = 15406732u.$ Again, for every unit in $u,$ the left-hand member of (4) becomes

$$\cdot 20000000 \uparrow' 1\,1,0,1,0,0,0,1, + \cdot 62500000 \downarrow, + 1\cdot 49875000$$
$$= 1\cdot 36806818 \text{ at least.}$$

And $\quad (1\cdot 32375000) \downarrow 0,3,3,0,8,1,5,5 = 1\cdot 36806818$

$\therefore$ the logarithm 28046962, is increased at least by the logarithm of $\downarrow 0,3,3,0,8,1,5,5, = 3293104,$ for every unit in $u,$ of the first position.

Hence the equation

$$28046962, + 3293104,u = 19118991, + 15406732,u$$

will give a convenient value of $u,$

$$\therefore \quad 12113628u = 8927971 \text{ gives } u = \cdot 07$$

Consequently, $\downarrow 0,7,u_3,$ may be taken as a convenient value for $\downarrow u,$

To avoid being misunderstood, these directions are given in detail, and the results carried far beyond the extent required to find a convenient value for $u,$; indeed, we might have taken $u, = {}^{\downarrow}1,$ without inconvenience.

$\updownarrow$ '0'7'27,0,2,0,7, is the reciprocal of $\downarrow$ 0,7,2,

$\uparrow$ '0'70,7,0,0,0,7, is the reciprocal of $\downarrow$ 0,7,

$\downarrow$ 0,7,u_3, being put for $\downarrow u_1$ in (4) the equation becomes

$$\downarrow, \{\cdot 18654360 (1 + [u_3) + \cdot 67008460 (1 \not\downarrow [u_3) + \cdot 49875\}$$

$$= \downarrow, \frac{(1\cdot 005)^{500 \downarrow 0,7,u_3,}}{(2)(5) \downarrow 0,7,u_3,}$$

$$= 30142692, \not\downarrow (267366433) [u_3 - \downarrow, (u_3,) ; \qquad (5).$$

$$\begin{array}{r} \cdot 18654360 \\ \cdot 67008460 \\ \cdot 49875000 \\ \hline \end{array}$$

$$\downarrow, (1\cdot 35537820) = 30408052,$$

For each unit in u_3 the right-hand member of (5) will be at least increased by 267366 and diminished by 99950, hence the increase for u_3 units will at least be $(267366 - 99950)u_3 = 167416u_3$. The left-hand member of (5) becomes

$$18635706 + \cdot 67075468 + \cdot 49875 = 1\cdot 35586174$$

for $u_3 = 1$ and $(1\cdot 35537820) \downarrow 3,5,6,7,1, = 1\cdot 35586174$;

consequently, the logarithm of the left-hand member is increased at least by 35671, for each unit in u_3;

$\therefore$ $30408052, + 35671 u_3$ being put $= 30142692, + 167416u_3$

gives $$u_3 = \frac{265360}{131745} = 2,01.$$

The reciprocal of $\downarrow$ 0,0,2,0,1, is $\updownarrow$ '0'0'20'12,'0'0

Again, to obtain greater accuracy, put $\downarrow$ 0,0,2,0,1,u_6, for u_3 in (5) and the equation becomes

$$\downarrow, \{\cdot 18616911 (1 \not\uparrow [u_6) + \cdot 67143215 (1 \not\downarrow [u_6) + \cdot 49875\}$$

$$= \downarrow, \frac{(1\cdot 005)^{500 \downarrow 0,7,2,0,1,u_6,}}{(2)(5) \downarrow 0,7,2,0,1,u_6,}$$

$$= 30479471, \not\downarrow (267904112) [u_6 - \downarrow, (u_6) ; \qquad (6).$$

$$\begin{array}{l} \cdot 18616911 \\ \cdot 67143215 \\ \cdot 49875 \\ \hline \end{array}$$

$$\downarrow, (1\cdot 35635126) = 30479822,$$

For each unit in u_6 the right-hand member of (6) will be at least increased by 168, therefore the increase for u_6 units will be $168 u_6 = (268 - 100) u_6$. The left-hand member of (6) becomes

$$\cdot 18616930 + \cdot 67143278 + \cdot 49875 = 1\cdot 35635208$$

for $\quad u_6 = 1, \quad$ and $\quad (1\cdot 35635126) \overset{7}{\downarrow} 6,1, = 1\cdot 35635208;$

consequently, the logarithm of the left-hand member of (6) is increased by 61, for each unit in u_6.

$$30479822, + 61 u_6 = 30479471 + 168 u_6$$

$$\therefore \quad u_6 = \frac{822}{\dot{1}07} \; 3,29$$

$$\therefore \quad x = 500 \downarrow 0,7,2,0,1,3,2,9, = 537\cdot 14748.$$

Ex. 19. *Let* 19 (a) *be the number of chances for the happening of an event in a single trial, and* 200 (b) *the number of chances for its failing; find how many trials* (x) *must be made to have an even chance that the event will happen* 3 (r) *times at least.*

$$\frac{a}{b} = \cdot 085 \; ; \quad \frac{a^2}{2b^2} = \cdot 0036125 \; ;$$

and the general equation becomes

$$x \left\{ \frac{1}{x} + \cdot 085 + \cdot 0036125 \, (x - 1) \right\} = \frac{(1\cdot 085)^x}{2}$$

which may be written

$$x \left\{ \frac{1}{x} + \cdot 0036125 x + \cdot 0813875 \right\} = \frac{(1\cdot 085)^x}{2}; \qquad (1).$$

The dual logarithm of 1·085 in the 12th position

$$= 81579986989,$$

$$\therefore \quad \downarrow,(1·085) = 8157999,$$

The value of x is situated between 30 and 40, since for 40, equation (1) becomes

$$40\left\{\frac{1}{40} + ·0036125(40) + ·0813875\right\} \text{ and } \frac{(1·085)^{40}}{(2)}$$

$$= \left\{\frac{1}{10} + ·578 + ·325551\right\} \text{ and } \frac{(1·085)^{40}}{(2)(10)}.$$

The logarithm of the left-hand member is

$$\downarrow,(1·00355000) = 354374,$$

but the logarithm of the right-hand member is greater

$$= 40\downarrow,(1·085) - \downarrow,(2) - \downarrow,(10) = 26746721,$$

When $x = 30\downarrow u,$ equation (1) becomes

$$30\downarrow u,\left\{\frac{1}{30\downarrow u,} + ·0036125(30\downarrow u,) + ·0813875\right\} = \frac{(1·085)^{30\downarrow u,}}{(2)}$$

$$\therefore \quad \left\{\frac{1}{5\downarrow u,} + ·65025\downarrow u, + ·488325\right\} = \frac{(1·085)^{30\downarrow u}}{(2)(5)\downarrow u,}; \qquad (2)$$

Neglecting $\downarrow u,$ and taking the logarithms of both sides of (2), we find

$$\downarrow,(1·338575) = 29160565,$$

and

$$30\downarrow,(1·085) - \downarrow,(10) = 244739961, - 230258509, = 14481452,$$

When	$x = 40$	354374,	is less than	26746721,
	$x = 30$	29160565,	is greater than	14481452

Hence the value of x lies between 30 and 40, limits sufficiently close to find the value of x to any required extent.

For each unit in $\downarrow u$, of the first position, the logarithm of the left-hand side of (2) is increased by 3439640, at least, and at the same time, the logarithm of the right-hand number of (2) is increased by (24473996, − 9531018,) = 14942878, at least.

$\therefore$ $29160565 + 3439640u$ being put $= 14481452 + 14942878u$ will point to a convenient value for $\downarrow u$,

$$u = \frac{14679113,}{11503238,} = 1{\cdot}2$$

$\therefore$ $\downarrow 1,2,u_8,$ may be put, for in (2) we obtain

$$\downarrow,\{{\cdot}17823564(1 \dagger [u_3) + {\cdot}72965203(1 \dagger [u_8) + {\cdot}488325\} =$$

$$\downarrow,\frac{(1{\cdot}085)^{30\downarrow 1,2,u_3,}}{(2)\,(5)\,(\downarrow 1,2,u_3)} = 32845564, \dagger\, 274625157\,[u_3 - \downarrow,u_3, ; \qquad (3)$$

For each unit in u_8 the logarithm of the left member of (3) is increased by 39500, and the right-hand member by 174675; hence, putting

$$33376334, + 59500u_8 = 32845564, - 174675u_8$$

$$\text{gives } u_8 = 3{\cdot}9$$

$$\therefore \quad x \text{ may be put } = 30\downarrow 1,2,5,9,^{u_1,}$$

the next step gives

$$x = 30\downarrow 1,2,3,9,1,6,6,0, = 33{\cdot}795352$$

This result, so easily found by the dual calculus, defied the combined skill of Laplace, De Moivre, the Bernoullis, and other writers on the theory of probabilities. The method here instituted will apply to equations of all dimensions equated to exponential equations.

Ex. 20. *Required the value of x in the equation*

$$7^x + 8^x = 9^x.$$

Ans. $x = 3{\cdot}94136679.$

The solution of this simple-looking question has heretofore defied the skill of mathematicians.

$$\left(\frac{7}{9}\right)^x + \left(\frac{8}{9}\right)^x = 1. \qquad \downarrow, \left(\frac{7}{9}\right) = {}'25131443 = {}'2'4'0'3'9'2'0'3\,'\uparrow$$

$$\downarrow, \left(\frac{8}{9}\right) = {}'11778304 = {}'1'1'2'3'7'1'1'8 \uparrow$$

Put $$x = n \downarrow u_1, u_2, \ldots\ldots$$

The given equation may be put under the form

$$({}'2'4'0'3'9'2'0'3 \downarrow)^{n+u_1,u_2,\ldots} + ({}'1'1'2'3'7'1'1'8 \uparrow)^{n+u_1,u_2,\ldots} = 1.$$

$$3\,({}'2'4'0 \uparrow) = {}'7'2 \uparrow = \cdot 468 \ldots\ldots \text{ nearly.}$$
$$3\,({}'1'1'2 \uparrow) = {}'3'3 \uparrow = \underline{\cdot 707} \ldots\ldots \text{ nearly.}$$
$$1\cdot 175 \qquad \text{too great.}$$

$$4\,({}'2'4'0 \uparrow) = {}'9'6 \uparrow = \cdot 364 \ldots\ldots \text{ nearly.}$$
$$4\,({}'1'1'2 \uparrow) = {}'4'4 \uparrow = \underline{\cdot 630} \ldots\ldots \text{ nearly.}$$
$$\cdot 994 \qquad \text{too small.}$$

Consequently x is less than 4, but greater than 3; however, we may commence operating with either of these numbers.

$$\begin{array}{r}
{}'11778304 \\
\underline{3} \\
\end{array}$$

$$'25131443 \quad \downarrow, (\cdot 702331962) = {}'35334912$$
$$\underline{3}$$
$$'75394329 = \downarrow, (\underline{\cdot 470507545})$$
$$1\cdot 1728 \ldots\ldots$$
$$\cdot 1728 \ldots\ldots \qquad \text{(A), Excess.}$$

$$('7|5\ldots)(\downarrow[u_1])('47\ldots) = \downarrow 3{\cdot}5\ 25u_1$$
$$('3|5\ldots)(\downarrow[u_1])('70\ldots) = \downarrow 2{\cdot}4\ 50u_1$$

$$\overline{\downarrow 5{\cdot}9\ 75u_1} = {\cdot}1728$$
$$+\ {\cdot}05975u_1$$

$$\therefore\quad = \frac{{\cdot}1728}{05975} = \downarrow 2, = \downarrow u_1$$

$\downarrow 2,$

$$\begin{array}{l}
\dot{'}7|5|3|9|4|3|2|9 \\ \hline
1|5|\dot{0}|7|8|8|6|6 \\
\quad\ \ 7|5|3|9|4|3 \\ \hline
'1\,5\,8\,3\,2\,8\,0\,9 \\ \hline
'9\,1\,2\,2\,7\,1\,3\,8
\end{array}
\begin{array}{l} \\ \\ \\ = '1'5'2'7'1'4'8'7\downarrow \\ \\ \end{array}$$

$\downarrow 2,$

$$\begin{array}{r}
'353349\dot{1}2 \\ \hline
7066982 \\
353349 \\ \hline
742033\dot{1} \\ \hline
'42755243
\end{array}
\begin{array}{l} \\ \\ \\ = '0'7'3'8'4'9'4'3\uparrow \\ \\ \end{array}$$

$$({\cdot}470507545)('1'5'2'7'1'4'8'7\uparrow) = {\cdot}401610974$$
$$({\cdot}702331962)('0'7'3'8'4'9'4'3\uparrow) = {\cdot}652\dot{1}\dot{0}32\dot{1}2$$

Then the corresponding results may be thus arranged :—

$$\begin{array}{rl}
'91227138 = & \downarrow,({\cdot}401610974) \\
 & \downarrow,({\cdot}652103212) = '42755243 \\ \hline
 & 1{\cdot}0537\,14186 \\
 & {\cdot}0537\ldots\ldots \quad \text{Excess, (B).}
\end{array}$$

$$('91|2\ldots)(\downarrow u_2)({\cdot}402) = 366624u_2$$
$$('42|7\ldots)(\downarrow u_2)({\cdot}652) = 278404u_2$$

$$\overline{645028u_2} = {\cdot}0537$$

$$\therefore\quad u_2 = 8,$$

$\overset{2}{\downarrow}\, 8,$

'91|22|71|38
7|29|81|71
|25|54|36
|51|09
|64
1
7 55 87 81 = '0'7'5'2'3'2'9'3 ↑
'98 78 59 19

$\overset{2}{\downarrow}\, 8,$

'42|75|52|43
3|42|04|19
|11|97|15
|23|94
|30
3 54 25 58 = '0'3'5'2'7'2'0'6 ↑
'46 29 78 01

('401610974) ('0'7'5'2'3'2'9'3 ↑) = ·372373023
('652103212) ('0'3'5'2'7'2'0'6 ↑) = ·629406473

The next convenient dual digit may be found by comparing these results.

'987859I9 = ↓, ('372373023)
↓, ('629406473) = '46297801
1·001779496
·001779... Excess, (C).

$('987|8 \ldots\ldots)(\downarrow u_3)('372 \ldots) = (368 \ldots)u_3$
$('462|9 \ldots\ldots)(\downarrow u_3)('629 \ldots) = (291 \ldots)u_3$
$659u_3 = 001779$ (C).

$\therefore\quad u_3 = 2,$

$\overset{3}{\downarrow}\, 2,$

'987|859|19
1|975|72
|99
1 976 71 '0'0'1'9'7'6'2'1 ↑
'989 835 90

$\overset{3}{\downarrow}\, 2,$

'462|978|01
|925|96
|46
926 42 = '9'2'6'4'2 ↑
'463 904 43

$$('372373023)\,('1'9'7'6'2'1\underset{3}{\uparrow}) = \cdot 371637659$$
$$('629406473)\quad ('9'2'6'4'2\underset{4}{\uparrow}) = \underline{\cdot 628823629}$$
$$1\cdot 000461288$$
$$\cdot 000461288 \quad \text{Excess, (D).}$$

We are now in a position to operate to find the next convenient dual digit, or u_4,

$$('9898| \ldots\ldots)(\downarrow u_4)(\cdot 3716) = 3678\ldots u_4$$
$$('4639| \ldots\ldots)(\downarrow u_4)(\cdot 6288) = \underline{2917\ldots u_4}$$
$$6595\,u_4 = 000461288 \qquad \text{(D).}$$

$$\therefore\quad u_4 = 7,$$

$$\overset{4}{\downarrow} 7, \qquad\qquad \overset{4}{\downarrow} 7,$$

'9898|3590	'46390443
6|9288	32473
21	10
6 9309 = '6'9'3'0'9 $\underset{4}{\uparrow}$	32483 = '5'2'4'8'5 $\uparrow$
'9905 2899	'46422926

$$(\cdot 371637659)\,('6'9'3'0'9\underset{4}{\uparrow}) = \cdot 371380158$$
$$(\cdot 628823629)\,('3'2'4'8'3\underset{4}{\uparrow}) = \underline{\cdot 628619392}$$
$$\cdot 999999550$$
$$\cdot 000000450 \quad \text{Defect, (P).}$$

The remaining digits may be found by common division, by retaining the last divisor (D).

$$\text{(D)}\ \ 65\dot{9}5 \qquad \begin{array}{r} 450 \\ \underline{396} \\ 54 \end{array} \qquad ('6'8\underset{7}{\uparrow}$$

$\therefore \quad x = '0'0'0'0'0'0'6'8 \uparrow 3 \downarrow 2,8,2,7,0,0,0,0,$

$$\| $$

$$3{\cdot}94139052$$

$$\therefore \quad 7^{3\cdot94139052} + 8^{3\cdot94139052} = 9^{3\cdot94139052}$$

Ex. 21. *Given* $3^x + 7^x = 16^x$ *to find* x.

Ans. $x = {\cdot}57812489.$

As in Example 1, it is evident that

$$\left(\frac{3}{16}\right)^x + \left(\frac{7}{16}\right)^x = 1.$$

$$\downarrow,\left(\frac{3}{16}\right) = '167397643$$

and $$\downarrow,\left(\frac{7}{16}\right) = '82667857$$

Put $$x = a \downarrow u_1, u_2, u_3, \ldots.$$

It may be soon confirmed that a may be put $= \frac{1}{2}$; for

$$\begin{array}{lll} 2\,)\,'167397643 & & 2\,)\,'82667857 \\ \hline \quad '83698822 = \downarrow,({\cdot}43301270) & & \\ \qquad \downarrow,({\cdot}66143783) = & & '41333929 \\ \hline \qquad 1{\cdot}09445\ldots & & \\ \qquad {\cdot}09445\ldots & & \text{Excess, (A).} \end{array}$$

To find a convenient value for u_1,

$$('83\ldots)(\downarrow u_1)({\cdot}43) = 3569u_1$$
$$('41\ldots)(\downarrow u_1)({\cdot}66) = 2706u_1$$
$$\overline{6275u_1 = {\cdot}09445;} \qquad \text{(A).}$$

$\therefore \quad u_1$ may be put $= 1$,

$\downarrow 1,$

'8	3	6	9	8	8	2	2	
	8	3	6	9	8	8	2	= '0'8'3'2'9'4'6'0 $\uparrow$
'92068704								

'4	1	3	3	3	9	2	9	
	4	1	3	3	3	9	3	= '0'4'1'1'3'2'0'7 $\uparrow$
'45467322								

(·43301270) ('0'8'3'2'9'4'6'0 $\uparrow$) = ·39824534
(·66143783) ('0'4'1'1'3'2'0'7 $\uparrow$) = ·63465535

1·0329 ....
·0329 .... Excess, (B).

To find a convenient value of u_2,

'92068704 = $\downarrow_1$ (·39824534)
$\downarrow_1$ (·63465535) = '45467322

('92 . . .) ($\downarrow u_2$) (·398) = $36616 u_2$
('45 . . .) ($\downarrow u_2$) (·634) = $28530 u_2$

$65146 u_2$ = ·0329 . . . (B).

Hence u_2 may be taken = 5,

$\overset{2}{\downarrow} 5,$

'92	06	87	04
4	60	34	35
	9	20	69
		9	21
			5
4	69	64	30 = '0'4'6'7'5'9'9'4 $\uparrow$
'96	76	51	34

$\overset{2}{\downarrow} 5,$

'45	46	73	22
2	27	33	66
	4	54	67
		4	55
			2
2	31	92	90 = '0'2'3'0'9'0'7'2 $\uparrow$
'47	78	66	12

(·39824534) ('0'4'6'7'5'9'9'4 $\uparrow$) = ·37997443
(·63465535) ('0'2'3'0'9'0'7'2 $\uparrow$) = ·62010525

1·00007968
·00007968 Excess, (C).

The remaining digits may be found by common division.

$$\begin{aligned} {}'96765134 &= \downarrow,(\cdot 37997443) \\ \downarrow,(\cdot 62010525) &= {}'47786612 \end{aligned}$$

$$\begin{aligned} ({}'967\ldots\ldots)(\not\downarrow[u\ldots)(\cdot 3799\ldots\ldots) &= 29636(u\ldots\ldots) \\ ({}'620\ldots\ldots)(\not\downarrow[u\ldots)(\cdot 4778\ldots\ldots) &= \underline{36746(u\ldots\ldots)} \\ &\ 66382(u\ldots\ldots) = \cdot 00007968 \end{aligned}$$

$$6638\ )\ 0000 \left| \begin{array}{l} 7968 \\ \underline{6638} \\ 1330 \end{array} \right. \qquad (\overset{4}{\downarrow}\, 1,2, = \overset{4}{\downarrow}\, u_v u_v$$

$$\begin{array}{l} \overset{4}{\downarrow}\ 1,2, \\ {}'9676|5134 \\ \quad\ \ \dagger\,\underline{}|\underline{} \\ \qquad\ \ |9677| \\ \qquad\ \ \ \underline{1935|} \\ \quad\ \ 11612 = {}'0'0'0'1'1'6'1'2\uparrow \\ \hline {}'9677\,6746 \end{array} \qquad\qquad \begin{array}{l} \overset{4}{\downarrow}\ 1,2, \\ {}'4778|6612 \\ \quad\ \ \dagger\,\underline{}|\underline{} \\ \qquad\ \ |4779| \\ \qquad\ \ \ \ \underline{956|} \\ \qquad\ \ 5735 = {}'0'0'0'0'5'7'3'5\uparrow \\ \hline {}'4779\,2347 \end{array}$$

$$\begin{aligned} (\cdot 37997443)({}'1'1'6'1'2\uparrow) &= \cdot 37993031 \\ (\cdot 62010525)(\ \ {}'5'7'3'5\uparrow) &= \underline{\cdot 62006968} \\ &\ \cdot 99999999 \end{aligned}$$

$$\therefore\ \ x = \tfrac{1}{2}\downarrow 1,5,0,1,2, = \cdot 57812489$$

Or, $\qquad 3^{\cdot 57812489} + 7^{\cdot 57812489} = 16^{\cdot 57812489}$

Ex. 22. *Given* $(42558\cdot 05712)^x + (52918\cdot 7469)^x = (60000\cdot)^x$, *to find* x.

Ans. $x = 3\cdot 21123237.$

Let $x = \beta \downarrow u_1, u_2, u_3, \ldots$

$$\frac{42558{\cdot}05712}{60000{\cdot}} = {\cdot}709300952\,; \quad \text{and} \quad \frac{52918{\cdot}7469}{60000{\cdot}} = {\cdot}881979115$$

Therefore,

$$('1'2'0'1'2'5'7'1 \uparrow)^{\beta \downarrow u_1, u_2, \ldots} + ('3'2'7'2'8'9'6'2 \uparrow)^{\beta \downarrow u_1, u_2, \ldots} = 1.$$

β may be put $= 3$, for if β be put $= 4$, the result is greater than 1.

$$\left.\begin{array}{l} 2('1'2 \uparrow) = '2'4 \uparrow \text{ nearly } = {\cdot}778 \ldots \\ 2('3'2 \uparrow) = '6'4 \uparrow \text{ nearly } = {\cdot}510 \ldots \end{array}\right\} \text{sum greater than 1.}$$

$$\left.\begin{array}{l} 3('1'2 \uparrow) = '3'6 \uparrow \text{ nearly } = {\cdot}364 \ldots \\ 3('3'2 \uparrow) = '9'6 \uparrow \text{ nearly } = {\cdot}686 \ldots \end{array}\right\} \text{sum greater than 1.}$$

But if β be taken $= 4$ then the sum would be less than 1. Hence β may be conveniently assumed $= 3$, yet, 2 or 4 may be put $= \beta$ and a correct value of x obtained. β may be either a multiple or submultiple, great, if $u_1, u_2, \ldots$ be small, and small if $u_1, u_2, \ldots$ be great. However, by one or two rough trials like those above instituted, a value may be given to β that will render its application convenient.

$$\begin{array}{ccccc} & ({\cdot}709300952)^x & + & ({\cdot}881979115)^x & = 1 \\ & \| & & \| & \\ & ('3'2'7'2'8'9'6'2 \uparrow)^x & + & ('1'2'0'1'2'5'7'1 \uparrow)^x & = 1 \\ & \| & & \| & \\ & '34347536 & & '12558691 & \\ & 3 & & 3 & \\ \hline \downarrow,({\cdot}356854885) = & '103042608 & & '37676073 = \downarrow,({\cdot}686080203) & \end{array}$$

$$\begin{array}{rl} {\cdot}686080203 & \\ {\cdot}356854885 & \\ \hline 1{\cdot}042935088 & \\ {\cdot}0429 \ldots\ldots & \text{Excess, (A).} \end{array}$$

('376 . . .) ($\downarrow$ [u) (·686 . . . = 2579 . . (u)
(1030 . . .) ($\downarrow$ [u)·(·356 . . . = 3677 . . (u)

6256 . . (u) = ·0429 . . (A).

It is evident that u_1 = 0; then u_2 = 6, for

```
6256u₂ = ·04|29 . . .   ( 6,
            3|75 . . .
```

```
   2                                  2
 ↓ 6,                               ↓ 6,
'37|67|60|73                     '10|30|42|60|8
-------------                    ---------------
  2|26|05|64|                       |61|82|55|6|
   | 5|65|14|                      1|54|56|4|
      | 7|54|                       | 2|06|1|
         | 6|                           | 1|5|
-------------                    ---------------
2 31 78 38 = '0'2'3'0'7'6'2'0 ↑  63 39 19 6 = '0'6'3'0'8'8'4'2 ↑
-------------                    ---------------
'39 99 39 11                     '10 93 81 80 4
```

(·686080203) ('0'2'3'0'7'6'2'0 ↑) = ·670360856
(·356854885) ('0'6'3'0'8'8'4'2 ↑) = ·334935266

'109381804 = ↓, (·334935266)
↓, (·670360856) = '39993911

1·005296122
·005296122 Excess, (B).

To find u_3.

('1093 . . . . .) ($\downarrow$ [u_3) (·3349) = 366 . . . u_3
('329 . . . . .) ($\downarrow$ [u_3)(·6703) = 268 . . . u_3

634u_3 = ·00529 (B).

∴ u_3 may be put = 8,

$$\begin{array}{r|r|r}
\multicolumn{3}{l}{\downarrow^{8} 8} \\
109 & 381 & 804 \\
\hline
 & 875 & 054 \\
 & 3 & 063 \\
 & & 6 \\
\hline
\multicolumn{3}{r}{878\,123} \\
\hline
\multicolumn{3}{l}{'110\,259\,927}
\end{array}
= '0'0'8'7'7'7'2'3 \uparrow
\qquad
\begin{array}{r|r|r}
\multicolumn{3}{l}{\downarrow^{8} 8,} \\
'399 & 939 & 11 \\
\hline
3 & 199 & 51 \\
 & 11 & 20 \\
 & & 2 \\
\hline
\multicolumn{3}{r}{3\,210\,73} \\
\hline
\multicolumn{3}{l}{'403\,149\,84}
\end{array}
= '0'0'3'2'0'9'2'3 \uparrow$$

$$\begin{array}{rr}
(\cdot 334935266)\,('0'0'8'7'7'7'2'3 \uparrow) = & \cdot 332006985 \\
(\cdot 670360856)\,('0'0'3'2'0'9'2'3 \uparrow) = & \cdot 668211852 \\
\hline
 & 1 \cdot 000218837 \\
 & \cdot 000218837 \quad \text{Excess, (C).}
\end{array}$$

$$\begin{aligned}
'110259927 &= \downarrow, (\cdot 332006985) \\
& \downarrow, (\cdot 668211852) = '40314984
\end{aligned}$$

To find u_4,

$$\begin{array}{rr}
('1102\,.\,.\,.\,.\,.)\,(\dagger\,[u_4])\,(\cdot 332) = & 365864u_4 \\
('403\,.\,.\,.\,.\,.)\,(\dagger\,[u_4])\,(\cdot 668) = & 269204u_4 \\
\hline
 & 635\ldots u_4 = \cdot 000218837
\end{array}$$

Then u_4, may be put $= 3$, and $u_5 = 4$, for

$$\begin{array}{r|l}
635\ )\ \cdot 0002 & 1883\ (\ 3,4, \\
1 & 905 \\
\hline
\multicolumn{2}{c}{2833} \\
\multicolumn{2}{c}{2540} \\
\hline
\end{array}$$

```
 4                                  4
↓ 3,4,                             ↓ 3,4,
'1102|5992|7                       '4031|4984
     ----------                     ----------
     |3307|8|                          1|2095|
          |3|                       ----------
------------    5                             5
      441 2| = ↓ 4,                     1613| = ↓ 4,
------------                          ------
     3749 3 = '0'0'0'3'7'4'9'3 ↑          1 3708 = '0'0'0'1'3'7'0'8 ↑
-----------                         ----------
'1102 97420                         '4033 2325
```

$(\cdot 332006985)\,('0'0'0'3'7'4'9'3\uparrow) = \cdot 331882524$
$(\cdot 668211852)\,('0'0'0'1'3'7'0'8\uparrow) = \cdot 668120257$

$1\cdot 000002881$ Excess, (D).

$('11029\ldots\ldots)\,(\downarrow[u_6])\,(\cdot 33188\ldots\ldots) = 36608\,u_6$
$('4033\ldots\ldots)\,(\downarrow[u_6])\,(\cdot 66812\ldots\ldots) = 26945\,u_6$

$63553\,u_6 = \cdot 000002881$ (D).

```
63553 ) ·000002|881 ( 4,
               2|542
               -----
                 339
                 318 ( 5,
                 ---
                  21
                  19 ( 3,
                  --
```

$\therefore\quad x = 3\downarrow 0,6,8,3,4,4,5,3, = 3\cdot 21123237$

Ex. 23. *Find the first eight figures of the continued product of the odd numbers* $1\cdot 3\cdot 5\cdot 7\cdot 9\ldots\ldots 505$.

In the Author's Work on the "Calculus of Form, a New Science," it is shown that

$$\downarrow,(1\cdot 3\cdot 5\cdot 7\ldots\ldots(2x-1)) = x\downarrow,(x) + (x+\tfrac{1}{2})\downarrow,(2) - 10^8 x.$$

In this example $x = 505$.

$$\downarrow,(505.) = 622455842,'9275$$

$$\downarrow,(2) = 69314718,056$$

$$\therefore\ x\downarrow,(x) = 314340200678,3875$$

$$35038589977,308 = (505\tfrac{1}{2})\downarrow,(2)$$

$$\dot{3}49378790655,6855$$

$$10^8 x = 50500000000,$$

$$298878790655,6855$$

This result divided by $\downarrow,(10) = 230258509,2994$ gives 1298 in the quotient with 3245585, remainder.

$$3245585, = \downarrow 0,3,2,6,0,5,8,6, = 1{\cdot}03298827.$$

Hence, the product will consist of 1299 places of figures, the first eight of which is 10329882.

Ex. 24. *In the curve whose equation is* $y = \dfrac{2}{\sqrt{\pi}}\epsilon^{-x^2}$ *find* y *when* $x = 3$. *And* x *when* $y = \dfrac{1}{10^8}$; ϵ *being equal to* $2{\cdot}718281828\ldots$

$\downarrow,(\epsilon) = 100000000,$

$$\downarrow,(y) = \downarrow,(2) - \tfrac{1}{2}\downarrow,(\pi) - x^2\downarrow,(\epsilon)$$

$$\downarrow,(2) = 69314718,055994533$$

$$\tfrac{1}{2}\downarrow,(\pi) = 57236494,292470008 = \downarrow,\left(\frac{2}{\sqrt{\pi}}\right)$$

$$12078223,763524525$$

$$\therefore\ \downarrow,(y) = 12078224, + '900000000 = '887921776$$

$$'887921776 = 33112261, - \downarrow,(10^4)$$

$$33112261, = \downarrow 3,4,5,3,9,3,2,5 = 1{\cdot}3925305$$

$$\therefore\ y = {\cdot}00013925305 \quad \text{when} \quad x = 3.$$

Again $\quad \downarrow, \left(\frac{1}{10^8}\right) = 12078224, - x^2(100000000,)$

$$\therefore \quad x^2 = \frac{1842068074, + 12078224,}{100000000,} = 18{\cdot}54146298$$

$$\therefore \quad x = 4{\cdot}3059799 \quad \text{when} \quad y = {\cdot}000000001$$

Ex. 25. *Find the area of the curve expressed by* $\frac{2}{\sqrt{\pi}}\int_0^x \epsilon^{-x^2} dx$ *and the differences for intervals of* $x = {\cdot}01$; $x = {\cdot}02$; $x = {\cdot}03$; &c.

First ordinate,

$$\begin{aligned} \downarrow, (y) &= 12078224, - ({\cdot}01)^2(100000000,) \\ &= 12068224, = \downarrow, 1,2,5,4,7,3,9,0, = \downarrow, (1{\cdot}12826630) \end{aligned}$$

$$\therefore \quad y = 1{\cdot}1282663 \quad \text{when} \quad x = {\cdot}01$$

Second ordinate,

$$\begin{aligned} \downarrow, (y) &= 12078224, - ({\cdot}02)^2(100000000,) \\ &= 12038224, = \downarrow, 1,2,5,1,7,3,9,0, = \downarrow, (1{\cdot}12792789) \end{aligned}$$

$$\therefore \quad y = 1{\cdot}12792789 \quad \text{when} \quad x = {\cdot}02$$

Third ordinate,

$$\begin{aligned} \downarrow, (y) &= 12078224, - ({\cdot}03)^2(100000000,) \\ &= 11988224, = \downarrow, 1,2,4,6,7,3,4,0, = \downarrow, (1{\cdot}12736404) \end{aligned}$$

$$\therefore \quad y = 1{\cdot}12736404 \quad \text{when} \quad x = {\cdot}03.$$

Fourth ordinate,

$$\begin{aligned} \downarrow, (y) &= 12078224, - ({\cdot}04)^2(100000000,) \\ &= 11918224, = \downarrow, 1,2,3,9,7,2,9,0 = \downarrow, (1{\cdot}12657515) \end{aligned}$$

$$\therefore \quad y = 1{\cdot}12657515 \quad \text{when} \quad x = {\cdot}04.$$

Fifth ordinate,

$$\downarrow,(y) = 12078224, - (\cdot 05)^2(100000000,)$$
$$= 11828224, = \downarrow, 1,2,3,0,7,2,9,0, = \downarrow,(1\cdot 12556174)$$
$$\therefore \quad y = 1\cdot 12556174 \quad \text{when} \quad x = \cdot 05.$$

Sixth ordinate,

$$\downarrow,(y) = 12078224, - (\cdot 06)^2(100000000,)$$
$$= 11718224, = \downarrow, 1,2,1,9,7,1,9,0, = \downarrow,(1\cdot 12432425)$$
$$\therefore \quad y = 1\cdot 12432425 \quad \text{when} \quad x = \cdot 06.$$

Seventh ordinate,

$$\downarrow,(y) = 12078224, - (\cdot 07)^2(100000000,)$$
$$= 11588224, = \downarrow, 1,2,0,6,7,1,4,0, = \downarrow,(1\cdot 12286359)$$

$$\begin{aligned} \therefore \quad y &= 1\cdot 12286359 \quad \text{when} \quad x = \cdot 07 \\ y &= 1\cdot 12118147 \quad \text{when} \quad x = \cdot 08 \\ y &= 1\cdot 11927617 \quad \text{when} \quad x = \cdot 09 \\ &\quad \text{\&c.} \qquad\qquad\qquad \text{\&c.} \end{aligned}$$

Two or three hundreds of these ordinates may be calculated in a few hours, and a table of corresponding areas formed for any range, as from $x = 0$ to $x = 2$ or $x = 3$ &c. When greater accuracy is required the intervals must be made less.

When $x = 0$, $\frac{2}{\sqrt{\pi}}\epsilon^{-x^2}$ becomes $\frac{2}{\sqrt{\pi}} = 1\cdot 1283792$ the ordinate at the origin.

$$\begin{array}{r} 1\ 1283792 \\ y_1 = 1\cdot 1282663 \\ \hline 2\ )\ 2\cdot 2566455 \\ \hline \end{array}$$

$1\cdot 1283228 \times \cdot 01 = \cdot 0112832$ area between 0 and y_1.

$$\begin{array}{r}
1·1282663 \\
1·1279279 \\
\hline
2\ )\ 2·2561942 \\
\hline
1·1280976
\end{array}$$

$\therefore$ ·0112810 = area between y_1 and y_2.

$$\begin{array}{r}
1·1279279 \\
1·1273640 \\
\hline
2\ )\ 2·2552919 \\
\hline
1·1276459
\end{array}$$

$\therefore$ ·0112765 = area between y_2 and y_3.

$$\begin{array}{r}
1·127364 \\
1·126575 \\
\hline
2\ )\ 2·253939 \\
\hline
1·126970
\end{array}$$

$\therefore$ ·0112697 = area between y_3 and y_4.

1·126575	1·125562	1·124324	1·122864	1·121181
1·125562	1·124324	1·122864	1·121181	1·119276
2) 2·252137	2) 2·249886	2) 2·247188	2) 2·244045	2) 2·240457
1·126069	1·124943	1·123594	1·122023	1·120229

In the practical application of the theory of probabilities, a table for the values of $\frac{2}{\sqrt{\pi}}\int_0^x \epsilon^{-x^2}\,dx = \text{A}$, for intervals of x each = ·01 is often employed. To construct such table by the dual calculus, requires but little numerical labour.

Specimen.

x	A	Diff.	
0·00	0·0000000	112833	The differences are calculated as shown above.
0·01	0·0112832	112810	
0·02	0·0225642	112765	
0·03	0·0338407	112697	
0·04	0·0451104	112607	
0·05	0·0563711	112494	
0·06	0·0676205	112359	
0·07	0·0788564	112202	
0·08	0·0900766	112023	
0·09	0·1012789		

Ex. 26. *What is the difference of the areas corresponding to the ordinates* 1·53 *and* 1·54, *in the curve of the previous Example?*

$$\downarrow, (y) = 12078224, - (1\cdot53)^2 \, (100000000,)$$

$$= 8246733, - \downarrow, (10)$$

$$8246733, = \downarrow, 0,8,2,8,6,5,6,9, = \downarrow, (1\cdot08596315)$$

$$\therefore \quad y = \cdot108596315 \text{ when } x = 1\cdot53$$

and $$y = \cdot105313068 \text{ when } x = 1\cdot54$$

$$2\;)\;\cdot213909383$$

$$\cdot106954692 \times \cdot01 = \cdot0010695..$$

$\therefore$ ·0010695 is the required difference.

Ex. 27. *Required the value of* $C = \sqrt{\dfrac{n}{2\pi x x_1}} \; e^{-\frac{nl^2}{2xx_1}}$ when

$n = 14000\cdot$; $x = 7200$; $x_1 = 6800$, and $l = 163$.

$$\downarrow,(C) = \downarrow,\left(\frac{n}{2\pi x x_1}\right)^{\frac{1}{2}} - \frac{n l^2}{2 x x_1}\ (100000000,)$$

$$\frac{n l^2}{2 x x_1} = \frac{(1{\cdot}63)^2\,(1{\cdot}4)\ 10^8}{2\ ({\cdot}72)\ ({\cdot}68)\ 10^8} = \frac{(1{\cdot}63)^2\,(1{\cdot}4)}{({\cdot}72)\ (1{\cdot}36)} = 3{\cdot}79867239$$

$$\frac{n}{2\pi x x_1} = \frac{14000{\cdot}}{2\pi\ (7200{\cdot})\ (6800{\cdot})} = \frac{1{\cdot}42973856}{10^4\ \pi}.$$

$$\therefore\quad \downarrow,\left(\frac{n}{2\pi x x_1}\right)^{\frac{1}{2}} = \text{'}499878930.$$

$$\therefore\quad \downarrow,(C) = \text{'}499878930 + \text{'}379867239$$

$$= 41287868, - \downarrow,\ (10^4)$$

$$41287868, = \downarrow,\ 4,3,1,7,8,7,4,7, = \downarrow,\ (1{\cdot}51116161).$$

$$\therefore\quad C = {\cdot}000151116161.$$

Ex. 28. *What is the area of the curve* $y = \frac{2}{\sqrt{\pi}}\ \epsilon^{-x^2}$ from $x = 3$ to $x = 4{\cdot}3$?

We have before shown how the ordinates may be almost instantly calculated from the formula

$$\downarrow,(y) = 12078224, - 10^8\,x^2.$$

x.	$-\ 10^8\,x^2$.	y.	
3·0	'900000000	·00013925	(1)
3·1	'961000000	·00007566	(2)
3·2	'1024000000	·00004030	(3)
3·3	'1089000000	·00002104	(4)
3·4	'1156000000	·00001076	(5)
3·5	'1225000000	·00000540	(6)
3·6	'1296000000	·00000265	(7)
3·7	'1369000000	·00000128	(8)
3·8	'1444000000	·00000061	(9)
3·9	'1521000000	·00000028	(10)
4·0	'1600000000	·00000013	(11)
4·1	'1681000000	·00000006	(12)
4·2	'1764000000	·00000002	(13)
4·3	'1849000000	·00000001	(14)

THOMAS SIMPSON'S RULE.

To the sum of the first and last, or extreme ordinates, add 4 times the sum of the 2d, 4th, 6th, &c., or even ordinates, and twice the sum of the 3d, 5th, 7th, &c., or odd ordinates, not including the extreme ones; the result, multiplied by $\frac{1}{3}$ the ordinates' equidistance, will be the area.

7566	4030	41488	four times even ordinates.
2104	1076	10894	twice the odd ordinates.
540	265	13925	first ordinate.
128	61	1	last ordinate.
28	13	———	
6	2	·00066308	
——	——	·1	ordinates' equidistance.
10372	5447	———	
4	2	3)·000066308	
——	——	———	
41488	10894	·000022103	area between $x = 3$ to $x = 4·3$.

∴ area between $x = 0$ and $x = 3$ } ·999977897

The area (A) of the curve whose equation is $y = \frac{2}{\sqrt{\pi}} \epsilon^{-x^2}$ may be represented by either of the following series:—

$$A = \frac{2}{\sqrt{\pi}} \left(x - \frac{1}{1} \frac{x^3}{3} + \frac{1}{1·2} \frac{x^5}{5} - \frac{1}{1·2·3} \frac{x^7}{7} + \dots \right); \quad (1). \text{ Convergent.}$$

$$A = 1 - \frac{\epsilon^{-x^2}}{x\sqrt{\pi}} \left(1 - \frac{1}{2x^2} + \frac{1·3}{2^2 x^4} - \frac{1·3·5}{2^3 x^6} + \frac{1·3·5·7}{2^4 x^8} - \dots \right); \quad (2). \text{ Divergent.}$$

The first of these formulæ may be employed when x is less than 2, and the second by the aid of continued fractions, when x is greater than 2; but to apply either (1) or (2) by common Arithmetic, when x is a compound number, almost amounts to an impossibility, except to obtain rough approximations.

Ex. 29. *What is the area of the curve whose equation is* $y = \frac{2}{\sqrt{\pi}} \epsilon^{-x^2}$ from $x = 2$ to $x = 3$?

x.	$-10^8 x^2$	y.	
2·0	’400000000	·02066985	(1)
2·1	’441000000	·01371565	(2)
2·2	’484000000	·00892216	(3)
2·3	’529000000	·00568902	(4)
2·4	’576000000	·00355515	(5)
2·5	’625000000	·00217828	(6)
2·6	’676000000	·00130805	(7)
2·7	’729000000	·00076992	(8)
2·8	’784000000	·00044421	(9)
2·9	’841000000	·00025121	(10)
3·0	’900000000	·00013925	(11)

```
1371565            892216
 568902            355515
 217828            130805
  76992             44421
  25121           -------
-------           1412957
2260408                 2
      4           -------
-------           2825914
9041632

 ·02066985
 ·09041632
 ·02825914
 ·00013925
 ---------
 ·13948456
        ·1
 ---------
3) ·013948456
 ----------
```

·004649485 = area from $x = 2·$ to $x = 3$.
·000022103 = area from $x = 3·$ to $x = 4·3$.

·004671588 = area from $x = 2·$ to $x - 4·3$.
Ar. Co. ·995328412 = area from $x = 0·$ to $x = 2$.

Ex. 30. *Suppose* 18 *boys are born to* 17 *girls, in which case out of* 14000 *births the most likely individual case is, that* 7200 *should be boys, and* 6800 *girls, what is the probability that the number of boys shall fall between* 7200 ± 163?

In Laplace's formula (Q) $x = 7200$ $x_1 = 6800$ $n = 14000$ $l = 163$ and $\pi = 3{\cdot}14159265\ldots$ &c. $\epsilon = 2{\cdot}718281828\ldots$ &c.

$$2\sqrt{\frac{n}{2\pi x x_1}}\int_0^l \epsilon^{-\frac{nl^2}{2xx_1}}\,dl + \sqrt{\frac{n}{2\pi x x_1}}\,\epsilon^{-\frac{nl^2}{2xx_1}}; \qquad \text{(Q)}.$$

In Example 27. $+\sqrt{\frac{n}{2\pi x x_1}}\,\epsilon^{-\frac{nl^2}{2xx_1}}$ is found to be $\cdot 000151116161$, and $\frac{nl^2}{2xx_1} = 3{\cdot}79867239$, which put $\doteq z^2$, then $z = 1{\cdot}94901832$.

$$\therefore \quad l = \sqrt{\frac{n}{2xx_1}} = z.$$

$$\therefore \quad \sqrt{\frac{n}{2xx_1}}\,dl = dz.$$

Then making z the independent variable,

$$2\sqrt{\frac{n}{2\pi x x_1}}\int_0^l \epsilon^{-\frac{nl^2}{2xx_1}}\,dl \text{ becomes } \frac{2}{\sqrt{\pi}}\int_0^z \epsilon^{-z^2}\,dz.$$

The area of the curve between $x = 1{\cdot}94901832$, and $z = 2{\cdot}00$ may be found as in previous examples.

When	$z = 2\cdot$	ordinate = ·02066985
	$z = 1{\cdot}94901832$	ordinate = ·02527625
	2) ·05098168	·04594610
Half diff.	·02549084	

$\cdot 0459461 \times \cdot 02549084 = \cdot 00117120 =$ area.

According to Example 29 the area between the ordinates for $z = 0$ to $z = 2\cdot$ was found to be

$$
\begin{array}{l}
= \cdot 99532841 \\
\ 00117120 \\
\hline
\ \cdot 99415721 = \text{area between 0 and } 1\cdot 94901832 \\
\ \cdot 00015111 \\
\hline
\ \cdot 99430832 = \text{the value of (Q).}
\end{array}
$$

It is therefore ·99430832 to ·00569168, or 175 to 1, that the number of male births shall be within the limits specified.

Suppose a perfectly flexible chain of uniform density and thickness to be suspended from two fixed points, A and B, and when in equilibrium to form the curve AOB; this curve

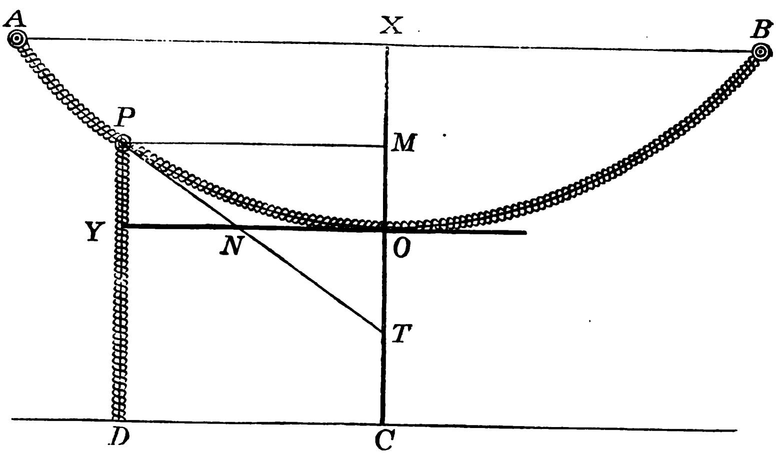


Fig. 4.

is termed the *catenary*. The equations to this useful curve being of a mixed and exponential kind, the calculations that have been made respecting it amount to little more than mere guess work.

Let O be the lowest point of the chain AOB; $OM = x$; $MP = y$; and the arc $OP = s$. Again, let v be the length of a portion of chain which is equal to the tension at O. If we suppose the part OP rigid, after it has assumed the form of equilibrium, it will evidently be supported in the same manner, and the tensions at O and P will be the same as when it was loose. OP is therefore kept at rest by three forces, namely, the tension at O acting in the direction of the tangent OY, the tension at P acting in the direction of PT, the tangent to the curve at the point P and the weight of the piece of chain OP acting in a vertical direction. Because the three forces just described are respectively parallel to the three sides of the triangle TMP, the forces being in equilibrium will be proportional to these sides.

$\therefore$ Weight of OP : tension at O :: TM : MP.

But ds, dy, and dx are respectively parallel to PT, PM, and MT; ds, dy, dx form a very small right-angled triangle at the point P.

$$\therefore \quad s : v :: dx : dy.$$

$$\therefore \quad \frac{dy}{dx} = \frac{v}{s}.$$

In every plane curve $ds^2 = dx^2 + dy^2$; hence in general

$$\frac{ds}{dx} = \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} \quad \therefore \quad \text{in the } \textit{catenary},$$

$$\frac{ds}{dx} = \frac{\sqrt{v^2 + s^2}}{s} \text{ or } dx = \frac{sds}{\sqrt{v^2 + s^2}}.$$

Taking the integral of this last equation, and observing that $s = 0$ when $x = 0$; we obtain

$$x + v - \sqrt{(v^2 + s^2)} \text{ or } s^2 = x^2 + 2vx \ldots\ldots (a).$$

Having determined v, (a) is the equation of the curve expressed by x and s as variables.

Again, because $\frac{dy}{dx} = \frac{v}{s} = \frac{v}{\sqrt{x^2 + 2vx}}$ which being integrated gives

$$\frac{y}{v} = \log_\epsilon \frac{x + v + \sqrt{x^2 + 2vx}}{v}$$

$$\therefore \quad \frac{y}{\epsilon^v} = \frac{v + x}{v} + \frac{\sqrt{x^2 + 2vx}}{v}; \; \ldots\ldots \quad (\beta).$$

ϵ being the base of the hyperbolic system of logarithms, (β) is the equation to curve between the variable co-ordinates x and y.

The equation (β) may be put in the form (γ) to render it convenient for dual calculation.

$$\epsilon^{\frac{y}{v}} = 1 + x\left(\frac{1}{v}\right) + \left\{\left(2x\right)\left(\frac{1}{v}\right)\left(1 + \frac{x}{2}\right)\left(\frac{1}{v}\right)\right\}^{\frac{1}{2}}; \qquad (\gamma).$$

Transposing $\frac{x + v}{v}$, and squaring both sides of (β) we obtain

$$\epsilon^{\frac{2y}{v}} - \epsilon^{\frac{y}{v}} \frac{x + v}{v} + \frac{(x + v)^2}{v^2} = \frac{x^2 + 2vx}{v^2}.$$

$$\therefore \quad x + v = \frac{v}{2}\left(\epsilon^{\frac{y}{v}} + \epsilon^{-\frac{y}{v}}\right);$$

and because (a), $s^2 = x^2 + 2vx$ or $s^2 = (x + v^2) - v^2$

$$\therefore \quad s = \frac{v}{2}\left(\epsilon^{\frac{y}{v}} - \epsilon^{-\frac{y}{v}}\right); \qquad (\delta).$$

which is the equation of the curve between the variables s and y; v being unknown but not variable in each particular inquiry.

Suppose t to be the length of a portion of the chain which is equal to the tension at any point P, then,

$$t : s :: \text{PT} : \text{TM} :: ds : dx$$

$$\therefore \quad sds = tdx;$$

Since $s^2 = x^2 + 2vx$, differentiating gives $sds = xdx + vdx$.

$$\therefore \quad tdx = 2dx + vdx \text{ and } t = x + v.$$

Suppose the tension at P to be balanced by means of PD, a portion of the chain passing over a pulley at P and hanging freely, then

$$\text{PD} = x + v = \text{OM} + v;$$

$\therefore$ YD = OC = v, which is evidently a constant quantity, although unknown. Hence, if the tension be supposed to be balanced by means of portions of the chain hanging over pulleys at points P, A, &c. the lower ends will be in the same horizontal line DC.

Example.

In a suspension bridge, let the central span AB, between the piers be 677·12 feet, the droop or deflection of the chain OX = 52·02 feet, the weight of the chain 365 tons; find the strains at the highest points A and B, and at the lowest point O.

$$\text{AX} : \text{XO} :: 1 : \frac{52\cdot02}{338\cdot56} = \cdot153650756.$$

Then if $\quad \text{AX} = y = 1;\ \text{OX} = x = \cdot15365076;$

$$\frac{x}{2} = \cdot07682538; \quad \text{and} \quad (2x)^{\frac12} = \cdot55434783.$$

Putting z for $\frac{1}{v}$ in equation (γ) it becomes

$$\epsilon^z = 1 + \cdot15365076z + \cdot55434783z^{\frac12}(1 + \cdot07682538z)^{\frac12}; \quad (1).$$

It will be hereafter shown that $\dfrac{6x}{3y^2 + x^2}$ is a rough limit to which z approaches;

$$\frac{6x}{3y^2 + x^2} = \cdot30490208 \text{ in the present Example.}$$

$\therefore \quad z$ may be put $= \cdot3025 \downarrow u, = \frac14 \downarrow 2, u,$ then $z^{\frac12} = \frac12 \downarrow 1 \ \dfrac{u}{2,}\cdot$

According to this design equation (1) becomes

$$\epsilon^{\cdot 3025 \downarrow u,} = [1 + \cdot 04647936 \downarrow u, + \cdot 30489130 \downarrow \frac{u}{2,} (1 + \cdot 02323968 \downarrow u,^{\frac{1}{2}}]; (2).$$

The dual logarithm of the left-hand member of (2), is 30250000, and the dual logarithm of the right-hand member is 30372259, when $u, = 0$. Hence it is evident from mere inspection, that it is convenient to suppose $u,$ a dual digit in the third position. It is further evident that for each unit in $u,$ the dual logarithm of the left-hand member of (2), will be at least increased by 30250, for

$$\begin{array}{r|r|l} 3\,0\,2 & 5\,0\,0 & 0\,0, \\ & 3\,0\,2 & 5\,0, \text{ for } \downarrow 0,0,1,. \end{array}$$

Again, for each unit in $\downarrow 0,0,u_3,$ the dual logarithm of the right-hand member of (2), will be at least increased by 15065,. Hence the following equation points out a convenient value for $u_3,$;

$$30250000, + 30250, u_3 = 30372259, + 15065, u_3$$

$$\therefore \quad u_3 = \frac{122259,}{15185,} = 8, \text{ nearly.}$$

Then $\downarrow 0,0,8, u_4,$ being substituted for $u_3,$ in equation (2), it becomes

$$\epsilon^{\cdot 30492847 \downarrow u_4,} = [1 + \cdot 04685350 \downarrow u_4,$$

$$+ \cdot 30611270 \downarrow \frac{u_4}{2}, (1 + \cdot 02342625 \downarrow u_4,)^{\frac{1}{2}}]; \quad (3).$$

When $u_4, = 0$, the dual logarithm of the left-hand member of (3) is 30492847, and the dual logarithm of the right-hand member is 30493068, and inspection shows that the next position to be occupied is the fifth.

Again, for each unit in $u_5,$ the logarithm of the left-hand member of (3) will be increased by 305, for

$$\begin{array}{r|l} 30492 & 847, \\ & 305, \text{ for } \downarrow 0,0,0,0,1. \\ \hline \end{array}$$

By substituting $\downarrow 0,0,0,0,1,$ for u_v in (3) its logarithm will be at least increased by 225, therefore, putting

$$30492847, + 305\,u, = 30493068, + 225\,u,$$

$$\text{gives } u = \frac{221}{80} = 2,7,$$

$$\therefore\quad z = {\cdot}3025 \downarrow 0,0,8,0,2,7,0,0, = \tfrac{1}{4} \downarrow 2,0,8,0,2,7,0,0,$$

$$\therefore\quad \frac{1}{z} = v = \text{reciprocal of } \tfrac{1}{4} \downarrow 2,0,8,0,2,7,0,0, = 2 \downarrow 5,1,8,0,0,0,5,9,$$

$$\therefore\quad v = 3{\cdot}2793685$$

$$\therefore\quad v \times 338{\cdot}56 = 1110{\cdot}263 \text{ feet.}$$

Tension at O = 1110·263 feet of chain.

Tension at A = 1162·283 feet of chain.

Although it is not the design of this work to discuss the limits of the roots of equations, we have on many occasions taken the most convenient limits. In the present inquiry we assumed that $\frac{6x}{3y^2 + x^2}$ approaches z, which is readily shown.

Because $$\frac{dy}{dx} = \frac{v}{\sqrt{x^2 + 2vx}} = \frac{v}{\sqrt{2vx}}\left(1 + \frac{x}{2v}\right)^{-\frac{1}{2}}$$

$$= \sqrt{\frac{v}{2x}}\left(1 - \frac{x}{4v} + \ldots\right)$$

$$\therefore\quad y = \sqrt{\frac{v}{2}}\left(2x^{\frac{1}{2}} - \frac{2}{3}\frac{x^{\frac{3}{2}}}{4v}\right) = \sqrt{2vx}\left(1 - \frac{x}{12v}\right)$$

from integrating and neglecting all the terms after the second.

$$\therefore\quad y^2 = 2vx\left(1 - \frac{x}{6v} + \frac{x^2}{144v^2}\right).$$

Again, in operating with the dual calculus $\frac{x^2}{144v}$ may be neglected, then $v = \frac{x^2 + 3y^2}{6x} \downarrow u_1, u_2, u_3, \ldots$ to any required degree of accuracy. The dual number $\downarrow u_1, u_2, u_3, \ldots$ makes good all the defects of $\frac{x^2 + 3y^2}{6x}$.

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